Journal of Functional Analysis 273 (2017) 3859–3874

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

On amenability and groups of measurable maps

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ARTICLE INFO

Article history:
Received 1 January 2017
Accepted 15 September 2017
Available online 22 September 2017
Communicated by G. Schechtman

Keywords:
Groups of measurable maps
Concentration of measure
Extreme amenability
Whirly actions

ABSTRACT

We show that if G is an amenable topological group, then the topological group $L^0(G)$ of strongly measurable maps from $([0,1], \lambda)$ into G endowed with the topology of convergence in measure is whirly amenable, hence extremely amenable. Conversely, we prove that a topological group G is amenable if $L^0(G)$ is.

1. Introduction

In this note we study measurable maps taking values in topological groups. Let us recall that a map $f\colon X \to Y$ from a compact Hausdorff space X carrying a regular Borel probability measure μ into a topological space Y is *strongly μ-measurable*[2] if, for every

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[2] Alternative terms used in the literature are *μ-measurable in the sense of Bourbaki* [3, p. 357], *Lusin μ-measurable* [26], or *μ-almost continuous* [2].

https://doi.org/10.1016/j.jfa.2017.09.011
0022-1236/

$\varepsilon > 0$, there exists a closed subset $A \subseteq X$ with $\mu(X \setminus A) \leq \varepsilon$ such that $f|_A : A \to Y$ is continuous. Evidently, if f is strongly μ-measurable, then it is μ-measurable, in the sense that $f^{-1}(B)$ is μ-measurable for every Borel $B \subseteq G$. In case Y is metrizable [2, Theorem 2B], the map f is strongly μ-measurable if and only if f is μ-measurable.

Let G be a topological group. Considering the Lebesgue measure λ on the real interval $[0,1]$, we define $L^0(G)$ to be the set of all λ-equivalence classes of strongly λ-measurable maps from $[0,1]$ into G. Equipping $L^0(G)$ with the group structure inherited from G and the topology of convergence in measure, we obtain a topological group. We refer to Section 4 for more details on $L^0(G)$.

The aim of this note is to prove the following, thus answering a question raised in [4].

Theorem 1.1. *Let G be a topological group. The following are equivalent.*

(1) *G is amenable.*
(2) *$L^0(G)$ is amenable.*
(3) *$L^0(G)$ is extremely amenable.*
(4) *$L^0(G)$ is whirly amenable.*

In particular, Theorem 1.1 provides a source of extremely amenable groups. The historically initial example, of a group with no nontrivial unitary representations, hence extremely amenable (since it is also amenable), was obtained by Herer and Christensen [9], as a group of measurable functions with values in the circle, however with a pathological submeasure in place of the Lebesgue measure.

For G the circle group, the extreme amenability of $L^0(G)$ (relative to the Lebesgue measure) was proved by Glasner, although only published many years later [5], and, as mentioned in the article, at the same time and independently it was proved by Furstenberg and Benjy Weiss, unpublished. (Of course, this group has plenty of unitary representations, recently classified by Solecki [27].) The result was later generalized to amenable locally compact groups G [18,20].

The proof of Theorem 1.1 (see Section 4) combines a recent result from [25] with a measure concentration argument. The application of measure concentration phenomena for proving extreme amenability was initiated by Gromov and Milman [8]. We note that for non-atomic submeasures the concentration technique cannot be used, because there is in general no concentration, so special combinatorial techniques must be applied instead, see Farah–Solecki [1] and Sabok [23], and the open questions therein. It would be interesting to see if the amenability criterion from [25] can be used to extend these combinatorial techniques to all amenable topological groups.

2. Preliminaries

In this section we fix some notation and briefly review some basic facts concerning uniform spaces, topological groups, and amenability.