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Loss of hyperbolicity and exponential growth for the viscous Burgers equation with complex forcing terms

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ABSTRACT

We investigate the phenomenon of the time-delay in the instabilities exhibited by the Cauchy problem for the complex viscous Burgers equation on the torus. Precisely, we see that the instantaneous amplification manifested by the solution of the inviscid equation is not observed when introducing a small viscous term in the system. What is more, we show that two distinct phases of the dynamics can be described, that is existence of a bounded solution in times of order one and, after that, an exponential growth in time. This phenomenon is ultimately related to a loss of hyperbolicity and to the subsequent transition to ellipticity for the inviscid problem. The key point of our analysis is a micro-local analysis of the symbol associated to the differential operator and the use of Gårding's inequality.

1. Introduction

In this paper we deal with the Cauchy problem for the one-dimensional Burgers equation with small viscosity and a complex forcing term

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$$\partial_t u + u\partial_x u - \varepsilon\partial_x^2 u = i, \quad u(x,0) = u_0(x), \tag{1.1}$$

where the variable $u(t,x)$ is complex valued with $t \in \mathbb{R}_+$ and $x \in \mathbb{T}$, and $0 < \varepsilon \ll 1$ can be seen as a small viscosity parameter. In particular, we are concerned in the description of a *time-delay in the instabilities* expected for the solution of such problem. This phenomenon can be summarize as follows: given a strongly unstable PDE (meaning that starting from a regular initial datum we observe an instantaneous amplification of the solution) if we add a small viscous term in the equation we observe the solutions to be bounded in times of order one before exhibiting an exponential growth. Precisely, the solutions of the equations under consideration exhibit a stable (observable) behavior for a certain time interval before they experience an instability in time.

Hence, there are two different time scales in the dynamics; the system appears to be perfectly stable for a long time, and it's only in the appropriate observation time that instabilities are detected.

The aim of this paper is to study such behavior for the solutions to (1.1) arisen from highly oscillating real initial data of amplitude $\mathcal{O}(1)$ and which depend on x through x/ε; in particular we show that such initial configurations generate solutions which are linearly increasing in time $\mathcal{O}(1)$, before exhibiting an exponential growth.

These results would validate the apparently naive approximation of (1.1) by the linear constant-coefficient equation

$$\partial_t v + it\partial_x v - \varepsilon\partial_x^2 v = 0, \tag{1.2}$$

for which the instability is manifested only after a delay in time, as can be easily shown in the Fourier side. Indeed, $\hat{v}(t,\xi)$ solves $\partial_t \hat{v} + (-t\xi + \varepsilon\xi^2)\hat{v} = 0$, so that

$$\hat{v}(t,\xi) = \hat{v}(0)\, e^{\xi\, t\, (\frac{t}{2} - \varepsilon\, \xi)},$$

and the exponential growth is recorded only for $t > 2\varepsilon\xi$. For example, if the initial datum is highly oscillating, i.e. $u_0(x) = u_0(x/\varepsilon)$, then the relevant frequencies are of order $\xi \sim 1/\varepsilon$ and the solution grows only after a delay in time of order $\mathcal{O}(1)$. What is more, if k_1 is the leading mode in the initial datum, say $u_0(x) = u_0(k_1 x/\varepsilon)$, then the solution grows only for $t > 2k_1$; in particular, the delay in time needed to observe an instability only depends on k_1. As we will see later on in this paper, this property is maintained invariant by the solutions to (1.1).

Before starting our analysis, we underline that equation (1.1) (and the relative problem of a delay in time for the amplification) has been already considered in [6]; here the authors succeed in proving bounds for the solution in times of order one. Also, they address the problem of the appearance of an instability for ulterior times, proving that an exponential growth of the solution indeed holds. However, such instability result holds true only for small data of amplitude $\mathcal{O}(\varepsilon^\alpha)$, $\alpha > \frac{1}{3}$; essentially, this smallness assumption allows to consider the nonlinear term as a small source in the equation (for more details,