Journal of Functional Analysis 273 (2017) 206–282

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Low lying spectral gaps induced by slowly varying magnetic fields

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A R T I C L E I N F O

Article history:
Received 10 May 2016
Accepted 4 April 2017
Available online 7 April 2017
Communicated by B. Schlein

Keywords:
Weakly variable magnetic fields
Spectral gaps
Landau Hamiltonian
Pseudo-differential operators

A B S T R A C T

We consider a periodic Schrödinger operator in two dimensions perturbed by a weak magnetic field whose intensity slowly varies around a positive mean. We show in great generality that the bottom of the spectrum of the corresponding magnetic Schrödinger operator develops spectral islands separated by gaps, reminding of a Landau-level structure.
First, we construct an effective Hofstadter-like magnetic matrix which accurately describes the low lying spectrum of the full operator. The construction of this effective magnetic matrix does not require a gap in the spectrum of the non-magnetic operator, only that the first and the second Bloch eigenvalues do not cross but their ranges might overlap. The crossing case is more difficult and will be considered elsewhere.
Second, we perform a detailed spectral analysis of the effective matrix using a gauge-covariant magnetic pseudo-differential calculus adapted to slowly varying magnetic fields. As an

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http://dx.doi.org/10.1016/j.jfa.2017.04.002

application, we prove in the overlapping case the appearance of spectral islands separated by gaps.

1. Introduction

In this paper we analyze the gap structure appearing at the bottom of the spectrum of a two dimensional periodic Hamiltonian which is perturbed by a magnetic field that is neither constant, nor vanishing at infinity, but which is supposed to have *'weak variation'*, in a sense made precise in Eq. (1.5) below. Our main purpose is to show the appearance of a structure of narrow spectral islands separated by open spectral gaps. We shall also investigate how the size of these spectral objects varies with the smallness and the weak variation of the magnetic field.

We therefore contribute to the mathematical understanding of the so called Peierls substitution at weak magnetic fields [34]; this problem has been mathematically analyzed by various authors (Buslaev [6], Bellissard [3,4], Nenciu [30,31], Helffer–Sjöstrand [17, 37], Panati–Spohn–Teufel [33], Freund–Teufel [13], De Nittis–Lein [11], Cornean–Iftimie–Purice [23,8]) in order to investigate the validity domain of various models developed by physicists like Kohn [26] and Luttinger [27]. An exhaustive discussion on the physical background of this problem can be found in [31].

1.1. Preliminaries

On the configuration space $\mathcal{X} := \mathbb{R}^2$ we consider a lattice $\Gamma \subset \mathcal{X}$ generated by two linearly independent vectors $\{e_1, e_2\} \subset \mathcal{X}$, and we also consider a smooth, Γ-periodic potential $V : \mathcal{X} \to \mathbb{R}$. The dual lattice of Γ is defined as

$$\Gamma_* := \left\{\gamma^* \in \mathcal{X}^* = \mathbb{R}^2 \mid \langle \gamma^*, \gamma\rangle/(2\pi) \in \mathbb{Z}\,,\ \forall \gamma \in \Gamma\right\}.$$

Let us fix an *elementary cell*:

$$E := \left\{y = \sum_{j=1}^{2} t_j e_j \in \mathbb{R}^2 \mid -1/2 \leq t_j < 1/2\,,\ \forall j \in \{1,2\}\right\}.$$

We consider the quotient group $\mathcal{X}/\Gamma$ that is canonically isomorphic to the 2-dimensional torus $\mathbb{T}$. The dual basis $\{e_1^*, e_2^*\} \subset \mathcal{X}^*$ is defined by $\langle e_j^*, e_k\rangle = (2\pi)\delta_{jk}$, and we have $\Gamma_* = \oplus_{j=1}^2 \mathbb{Z} e_j^*$. We define $\mathbb{T}_* := \mathcal{X}^*/\Gamma_*$ and E_* by

$$E_* := \left\{\theta = \sum_{j=1}^{2} t_j e_j^* \in \mathbb{R}^2 \mid -1/2 \leq t_j < 1/2\,,\ \forall j \in \{1,2\}\right\}.$$

Consider the differential operator $-\Delta + V$, which is essentially self-adjoint on the Schwartz set $\mathscr{S}(\mathcal{X})$. Denote by H^0 its self-adjoint extension in $\mathcal{H} := L^2(\mathcal{X})$. The map