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Module tensor product of subnormal modules need not be subnormal

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ABSTRACT

Let $\kappa : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be a diagonal positive definite kernel and let $\mathscr{H}_\kappa$ denote the associated reproducing kernel Hilbert space of holomorphic functions on the open unit disc $\mathbb{D}$. Assume that $zf \in \mathscr{H}$ whenever $f \in \mathscr{H}$. Then $\mathscr{H}$ is a Hilbert module over the polynomial ring $\mathbb{C}[z]$ with module action $p \cdot f \mapsto pf$. We say that $\mathscr{H}_\kappa$ is a subnormal Hilbert module if the operator $\mathscr{M}_z$ of multiplication by the coordinate function z on $\mathscr{H}_\kappa$ is subnormal. N. Salinas (1988) [11] asked whether the module tensor product $\mathscr{H}_{\kappa_1} \otimes_{\mathbb{C}[z]} \mathscr{H}_{\kappa_2}$ of subnormal Hilbert modules $\mathscr{H}_{\kappa_1}$ and $\mathscr{H}_{\kappa_2}$ is again subnormal. In this regard, we describe all subnormal module tensor products $L^2_a(\mathbb{D}, w_{s_1}) \otimes_{\mathbb{C}[z]} L^2_a(\mathbb{D}, w_{s_2})$, where $L^2_a(\mathbb{D}, w_s)$ denotes the weighted Bergman Hilbert module with radial weight

$$w_s(z) = \frac{1}{s\pi}|z|^{\frac{2(1-s)}{s}} \quad (z \in \mathbb{D},\ s > 0).$$

In particular, the module tensor product $L^2_a(\mathbb{D}, w_s) \otimes_{\mathbb{C}[z]} L^2_a(\mathbb{D}, w_s)$ is never subnormal for any $s \geq 6$. Thus the answer to this question is no.

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1. Introduction

Let $\mathscr{H}$ be a reproducing kernel Hilbert space of holomorphic functions defined on the unit disc $\mathbb{D}$ such that $zf \in \mathscr{H}$ whenever $f \in \mathscr{H}$. Thus the linear operator $\mathscr{M}_z$ of multiplication by the coordinate function z on $\mathscr{H}$ is bounded. This allows us to realize $\mathscr{H}$ as a *Hilbert module* over the polynomial ring $\mathbb{C}[z]$ with module action given by

$$(p, f) \in \mathbb{C}[z] \times \mathscr{H} \longmapsto p(\mathscr{M}_z)f \in \mathscr{H}.$$

Following [9], we say that the Hilbert module $\mathscr{H}$ is *contractive* if the operator norm of the multiplication operator $\mathscr{M}_z$ is at most 1. Further, we say that $\mathscr{H}$ is *subnormal* if $\mathscr{M}_z$ is subnormal, that is, $\mathscr{M}_z$ has a normal extension in a Hilbert module containing $\mathscr{H}$ (refer to [7] for a comprehensive account on subnormal operators). By Agler's Criterion [1, Theorem 3.1], $\mathscr{H}$ is a contractive subnormal Hilbert module if and only if for every $f \in \mathscr{H}$, $\phi_f(n) = \|z^n f\|^2$ $(n \in \mathbb{N})$ is completely monotone for every $f \in \mathscr{H}$. Recall from [5] that $\phi : \mathbb{N} \to (0, \infty)$ is *completely monotone* if

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \phi(n+j) \geq 0 \text{ for all } m, n \in \mathbb{N}.$$

Remark 1.1. Note that if ϕ is a completely monotone sequence then so is ψ_m for any $m \in \mathbb{N}$, where $\psi_m(n) = \phi(m+n)$ $(n \in \mathbb{N})$.

We further note that, as a consequence of Hausdorff's solution to the Hausdorff's moment problem [5, Chapter 4, Proposition 6.11], $\mathscr{H}$ is a contractive subnormal Hilbert module if and only if for every unit vector $f \in \mathscr{H}$, $\{\|z^n f\|^2\}_{n \in \mathbb{N}}$ is a *Hausdorff moment sequence*, that is, there exists a unique probability measure μ_f supported in $[0, 1]$ such that

$$\|z^n f\|^2 = \int\limits_{[0,1]} t^n d\mu_f \ (n \in \mathbb{N}).$$

In this text, we are primarily interested in the following one parameter family of subnormal Hilbert modules.

Example 1.2. For a real number $s > 0$, consider the Hilbert space $L^2_a(\mathbb{D}, w_s)$ of holomorphic functions defined on the open unit disc $\mathbb{D}$ which are square integrable with respect to the weighted area measure $w_s\, dA$ with radial weight function

$$w_s(z) = \frac{1}{s\pi} |z|^{\frac{2(1-s)}{s}} \ (z \in \mathbb{D}).$$

Then $L^2_a(\mathbb{D}, w_s)$ is a Hilbert module over the polynomial ring $\mathbb{C}[z]$ (refer to [10] for the basic theory of weighted Bergman spaces). Since $L^2_a(\mathbb{D}, w_s)$ is a closed subspace of