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# Uniqueness properties for discrete equations and Carleman estimates

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Using Carleman estimates, we give a lower bound for solutions to the discrete Schrödinger equation in both dynamic and stationary settings that allows us to prove uniqueness results, under some assumptions on the decay of the solutions.


## 1. Introduction

The aim of this paper is to continue the study started in [9,10] to prove uniqueness properties for functions $u \in C^1([0,1], \ell^2(\mathbb{Z}^d))$ which satisfy the property

$$|i\partial_t u_j + \Delta_d u_j| \leq |V_j u_j|, \quad t \in [0,1],\ j \in \mathbb{Z}^d, \tag{1}$$

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with bounded potential $V$, under the assumptions that the function $u$ has fast decay at times $t = 0$ and $t = 1$. Here $\Delta_d$ stands for the discrete Laplace operator

$$\Delta_d u_j = \sum_{k=1}^{d} (u_{j+e_k} + u_{j-e_k} - 2u_j), \quad j \in \mathbb{Z}^d,$$

where $e_k$ is the standard basis of $\mathbb{R}^d$.

In particular, all the results we give can be written in terms of solutions to the discrete Schrödinger equation

$$i\partial_t u_j + \Delta_d u_j + V_j u_j = 0, \quad t \in [0,1], \ j \in \mathbb{Z}^d. \tag{2}$$

In the continuous case, these results are related to the Hardy uncertainty principle for the Fourier transform:

$$|f(x)| \leq Ce^{-|x|^2/\beta^2}, \quad |\hat{f}(\xi)| \leq Ce^{-4|\xi|^2/\alpha^2}, \text{ and } 1/\alpha\beta > 1/4 \Longrightarrow f \equiv 0.$$
$$\text{If } 1/\alpha\beta = \frac{1}{4}, \text{ then } f(x) = ce^{-|x|^2/\beta^2}.$$

The relation comes from the fact that basically the solution to the free Schrödinger equation, $i\partial_t u + \Delta u = 0$, has the same size as the Fourier transform of an appropriately modulated initial datum, so then the Hardy uncertainty principle can be stated, in an $L^2$ setting, as follows:

$$\|e^{\alpha|x|^2} u(0)\|_{L^2(\mathbb{R}^d)} + \|e^{\beta|x|^2} u(1)\|_{L^2(\mathbb{R}^d)} < +\infty, \quad \alpha\beta > \frac{1}{16} \Rightarrow u \equiv 0.$$

The classical proof of the Hardy uncertainty principle is based on complex analysis arguments (Phragmén–Lindelöf principle and properties of entire functions), but in the dynamical context there is a series of papers, [2,3,5,6,8], where the authors prove the Hardy uncertainty principle using real variable methods. Furthermore, not only do they prove their results for the free evolution, but they also include a potential term $Vu$ to the Schrödinger equation, under some size constraints for the potential $V$ but without any regularity assumption on it. The main techniques in the proof of their results are log-convexity properties for solutions with Gaussian decay and Carleman estimates.

In the discrete setting, the first thing we have to understand is how to replace the Gaussian decay, so in [9] we give an analogous version of the Hardy uncertainty principle by using complex analysis arguments that suggests that the discrete version of the Gaussian we have to consider is the product of modified Bessel functions, given by the following integral representation,

$$I_m(x) = \frac{1}{\pi} \int_0^{\pi} e^{z\cos\theta} \cos(m\theta), \ m \in \mathbb{Z}.$$