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An unconditionally saturated Banach space with the scalar-plus-compact property

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A B S T R A C T

We construct a Bourgain–Delbaen $\mathscr{L}_\infty$-space $\mathfrak{X}_{Kus}$ with structure that is strongly heterogeneous: any bounded operator on $\mathfrak{X}_{Kus}$ is a compact perturbation of a multiple of the identity, whereas the space $\mathfrak{X}_{Kus}$ is saturated with unconditional basic sequences.

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1. Introduction

J. Bourgain and F. Delbaen presented in [8] a brilliant method of constructing $\mathscr{L}_\infty$-spaces with peculiar structure. Their method relies on a careful choice of an increasing sequence of finite dimensional subspaces $(F_n)_n$ of $\ell_\infty(\Gamma)$, with countably infinite Γ and each F_n uniformly isomorphic to $\ell_\infty^{dim F_n}$. A suitable choice of $(F_n)_n$ guarantees that the space $\overline{\cup_n F_n}$ is an $\mathscr{L}_\infty$-space with no unconditional basis. The Bourgain–Delbaen example contains no isomorphic copy of c_0, answering an old problem in the theory of $\mathscr{L}_\infty$-spaces. Later R. Haydon [15] proved that this space is saturated with reflexive ℓ_p spaces and introduced the notation used nowadays. The Bourgain–Delbaen method was used to construct Banach spaces that solved several other long-standing conjectures on the structure of Banach spaces and showed that one may not hope for an ordinary classification of $\mathscr{L}_\infty$-spaces as it happens in the $C(K)$-spaces case, see [1–3,10]. We refer to [7] and [8] for the properties of the classical Bourgain–Delbaen spaces.

In [2] a general Bourgain–Delbaen-$\mathscr{L}_\infty$-space is defined and the authors show a remarkable fact that any separable $\mathscr{L}_\infty$-space is isomorphic to such a space. We recall from [2] that a BD-$\mathscr{L}_\infty$-space is a space $\mathfrak{X} \subset \ell_\infty(\Gamma)$, with Γ countable, associated to a sequence $(\Gamma_q, i_q)_{q\in\mathbb{N}}$, where $(\Gamma_q)_q$ is an increasing sequence of finite sets with $\Gamma = \cup_{q\in\mathbb{N}}\Gamma_q$ and $(i_q)_q$ are uniformly bounded compatible extension operators $i_q : \ell_\infty(\Gamma_q) \to \ell_\infty(\Gamma)$, i.e. $i_q(x)|_{\Gamma_q} = x$ and $i_q(x) = i_p(i_q(x)|_{\Gamma_p})$ for any $q < p$ and $x \in \ell_\infty(\Gamma_q)$. The space $\mathfrak{X} = \mathfrak{X}_{(\Gamma_q,i_q)_q}$ is defined as $\mathfrak{X} = \overline{\langle d_\gamma : \gamma \in \Gamma\rangle}$, where d_γ is given by $d_\gamma = i_q(e_\gamma)$, with q chosen so that $\gamma \in \Gamma_q \setminus \Gamma_{q-1}$. An efficient method of defining particular examples of BD-$\mathscr{L}_\infty$-spaces as quotients of canonical BD-$\mathscr{L}_\infty$-spaces was given in [5]. The authors proved that given a BD-$\mathscr{L}_\infty$-space $\mathfrak{X} \subset \ell_\infty(\Gamma)$ any so-called self-determined set $\Gamma' \subset \Gamma$ produces a further $\mathscr{L}_\infty$-space $Y = \overline{\langle d_\gamma : \gamma \in \Gamma \setminus \Gamma'\rangle}$ and a BD-$\mathscr{L}_\infty$-space $\mathfrak{X}/Y$, with the quotient map defined by the restriction of Γ to Γ'.

S.A. Argyros and R. Haydon in [3] used the Bourgain–Delbaen method in order to produce an $\mathscr{L}_\infty$-space $\mathfrak{X}_{AH}$ which is hereditary indecomposable (HI) i.e. contains no closed infinitely dimensional subspace which is a direct sum of further two closed infinitely dimensional subspaces (in particular the space $\mathfrak{X}_{AH}$ admits no unconditional basic sequence), and with dual isomorphic to ℓ_1. Moreover, using in an essential way the local unconditional structure imposed by the $\ell_\infty^{dim F_n}$-spaces they proved that the space $\mathfrak{X}_{AH}$ has the scalar-plus-compact property i.e. every bounded operator on the space is of the form $\lambda I + K$, with K compact and λ scalar.

Although it readily follows that there does not exist a Banach space with an unconditional basis and the scalar-plus-compact property, the latter property does not exclude rich unconditional structure inside the space. This is witnessed in [1], where it was shown that, among other spaces, any separable and uniformly convex Banach space embeds into an $\mathscr{L}_\infty$-space with the scalar plus compact property. Therefore, a naturally arising question is whether there exists a Banach space with the scalar-plus-compact property that is saturated with unconditional basic sequences.