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Derivations with values in ideals of semifinite von Neumann algebras

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ABSTRACT

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ and let $\mathcal{A}$ be an arbitrary C^*-subalgebra of $\mathcal{M}$. Assume that E is a fully symmetric function space on $(0, \infty)$ having Fatou property and order continuous norm and $E(\mathcal{M}, \tau)$ is the corresponding symmetric operator space. We prove that every derivation $\delta : \mathcal{A} \to \mathcal{E}(\mathcal{M}, \tau) := E(\mathcal{M}, \tau) \cap \mathcal{M}$ is inner, strengthening earlier results by Kaftal and Weiss [28]. In the case when $\mathcal{M}$ is a semifinite non-finite factor, we show that our assumptions on $E(0, \infty)$ are sharp.

1. Introduction

Let $\mathcal{N}$ be a von Neumann algebra and let J be an $\mathcal{N}$-bimodule. A derivation $\delta : \mathcal{N} \to J$ is a linear mapping satisfying $\delta(XY) = \delta(X)Y + X\delta(Y)$, $X, Y \in \mathcal{N}$. In particular, if

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$K \in J$, then $\delta_K(X) := KX - XK$ is a derivation. Such derivations implemented by elements in J are called *inner*. One of the classical problems in operator algebra theory is the question whether every derivation from $\mathcal{N}$ into J is automatically inner. The classical result in this direction is that in the special case, when the $\mathcal{N}$-bimodule J coincides with algebra $\mathcal{N}$, every derivation $\delta : \mathcal{N} \to \mathcal{N}$ is necessarily inner [39,40]. At the same time, when one considers more general $\mathcal{N}$-bimodules, there are examples of non-inner derivations from $\mathcal{N}$ into some ideals of a von Neumann algebra $\mathcal{M}$ with $\mathcal{N} \subset \mathcal{M}$ (see e.g. [37]).

Johnson and Parrott [25] considered the special case, when the larger algebra $\mathcal{M}$ coincides with the algebra $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ and the bimodule J is the ideal $K(\mathcal{H})$ of all compact operators on $\mathcal{H}$. In that paper, the authors proved that if $\mathcal{N}$ is an abelian von Neumman subalgebra of $\mathcal{M}$, then every derivation δ from $\mathcal{N}$ to $K(\mathcal{H})$ is automatically inner. As an easy consequence, they were also able to treat the case when $\mathcal{N}$ has no certain type II_1 factors as direct summands. The remaining case, when $\mathcal{N}$ is a von Neumann algebra of type II_1 was later resolved by Popa in [36].

Motivated by [25], derivations relative to semifinite von Neumann algebras have been widely investigated (see e.g. [5,1,37,28] and [12, Chapter 10]). Accompanied with the rapid development of the theory of symmetrically normed spaces (see e.g. [29,31,15,42]), interesting results concerning derivations with values in symmetrically (quasi-)normed spaces are also established in [4,3,7–9].

In [28], Kaftal and Weiss studied the derivation problem in the setting when $\mathcal{M}$ is an arbitrary semifinite von Neumann algebra with a semifinite faithful normal trace τ and $\mathcal{N}$ is a unital abelian or properly infinite von Neumann subalgebra of $\mathcal{M}$. Under this hypothesis, it is proved in [28, Theorem 14] that every derivation $\delta : \mathcal{N} \to \mathcal{L}_p(\mathcal{M}, \tau) := L_p(\mathcal{M}, \tau) \cap \mathcal{M}$, $1 \le p < \infty$, is inner, where $L_p(\mathcal{M}, \tau)$ is the noncommutative L_p-space relative to τ (see [35] and Section 2 below). However, the question whether every derivation from an arbitrary von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ into $\mathcal{L}_p(\mathcal{M}, \tau)$, $1 \le p < \infty$, is inner was left unresolved in that paper. Our main objective in the present paper is to answer this question in the affirmative. Furthermore, rather than just studying derivations with values in $\mathcal{L}_p(\mathcal{M}, \tau)$, $1 \le p < \infty$, we prove that every derivation from an arbitrary C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}$ into ideal $E(\mathcal{M}, \tau) \cap \mathcal{M}$ is inner, whenever $E(\mathcal{M}, \tau)$ is the symmetric operator space corresponding to a fully symmetric function space $E(0, \infty)$ on $(0, \infty)$ having Fatou property and order continuous norm, extending earlier results by Kaftal and Weiss [28, Theorem 14 and Corollary 15]. In Theorem 3.8, we demonstrate the sharpness of our assumptions on $E(0, \infty)$.

To conclude this section we note that throughout this paper we denote symmetric space (of possible unbounded operators) on $\mathcal{M}$ by $(E(\mathcal{M}, \tau), \|\cdot\|_E)$ (see Section 2 below), while the corresponding ideal in $\mathcal{M}$ by $\mathcal{E}(\mathcal{M}, \tau) = E(\mathcal{M}, \tau) \cap \mathcal{M}$. The latter ideal is equipped with the norm $\|\cdot\|_E$, however no assumption on completeness of $\mathcal{E}(\mathcal{M}, \tau)$ with respect to $\|\cdot\|_E$ is imposed.