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The second mixed projection problem and the projection centroid conjectures

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ABSTRACT

We provide partial answers to the open problems 4.5, 4.6 of [3] and 12.9 of [12] regarding the classification of fixed points of the second mixed projection operator and iterates of the projection and centroid operators.

1. Introduction

The setting of this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A compact convex subset of $\mathbb{R}^n$ with non-empty interior is called a *convex body*. The set of convex bodies in $\mathbb{R}^n$ is denoted by K^n. Write K^n_e for the set of origin-symmetric convex bodies. Also, write B^n and S^{n-1} for the unit ball and the unit sphere of $\mathbb{R}^n$. Moreover, ω_k denotes the volume of B^k.

The support function of $K \in \mathrm{K}^n$, $h_K : S^{n-1} \to \mathbb{R}$, is defined by

$$h_K(u) = \max_{x \in K} x \cdot u.$$

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Assume $K \in \mathrm{K}^n$, $n \geq 2$. The ith projection body $\Pi_i K$ of K is the origin-symmetric convex body whose support function, for $u \in S^{n-1}$, is given by

$$h_{\Pi_i K}(u) = \frac{1}{2} \int\limits_{S^{n-1}} |u \cdot x| dS_i(K, x),$$

where $S_i(K, \cdot)$ is the mixed area measure of i copies of K and $n-1-i$ copies of B^n; see [3, Section A3]. Note that Π_{n-1} coincides with the usual projection operator Π. We refer the reader to [11], especially Proposition 2, regarding the importance of classification of solutions to $\Pi_i^2 K = cK + \vec{v}$, where c is a positive constant and $\vec{v}$ is a vector. Let us remark that $\Pi_i B^n = \omega_{n-1} B^n$ and $\Pi_i^2 B^n = \omega_{n-1}^n B^n$. [3, Problems 4.6] and [12, Problems 12.7] ask which convex bodies K are such that $\Pi_i^2 K$ is homothetic to K. The case $i = n-1$ has received partial answers; see [9,16,20]. Schneider [17] deals with the case $i = 1$ and proves origin-centered balls are the only solutions to $\Pi_1^2 K = cK$. Grinberg and Zhang [5] provide an alternative path to this result. Motivated by the work of Fish, Nazarov, Ryabogin and Zvavitch [2] where the idea of considering the iteration problems locally was first considered, here we prove local uniqueness theorems for fixed points of the second mixed projection operators for $1 < i < n-1$:

Theorem 1.1. *Suppose $n \geq 3$ and $1 < i < n-1$. There exists $\varepsilon > 0$ with the following property. If a convex body K satisfies $\Pi_i^2 K = cK + \vec{v}$ for some $c > 0$ and $\vec{v} \in \mathbb{R}^n$, and $\|h_{\lambda K + \vec{a}} - 1\|_{C^2} \leq \varepsilon$ for some $\lambda > 0$ and $\vec{a} \in \mathbb{R}^n$, then K is a ball.*

A set K in $\mathbb{R}^n$ is called star-shaped if it is non-empty and if $[0, x] \subset K$ for every $x \in K$. For a compact star-shaped set K, the radial function ρ_K is defined by

$$\rho_K(x) = \max\{\lambda \geq 0; \lambda x \in K\}, \quad x \in \mathbb{R}^n - \{0\}.$$

A compact star-shaped set with a positive continuous radial function is called a star body.

The polar body, K^*, of a convex body K with the origin in its interior is the convex body defined by

$$K^* = \{x \in \mathbb{R}^n; x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It follows from the definition that $\rho_{K^*} = \frac{1}{h_K}$ on S^{n-1}.

The centroid body of a star body K is an origin-symmetric convex body whose support function, for $u \in S^{n-1}$, is given by

$$h_{\Gamma K}(u) = \int\limits_{S^{n-1}} |u \cdot x| \rho_K^{n+1}(x) dx.$$