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Nodal intersections for random waves against a segment on the 3-dimensional torus

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A B S T R A C T

We consider random Gaussian eigenfunctions of the Laplacian on the three-dimensional flat torus, and investigate the number of nodal intersections against a straight line segment. The expected intersection number, against any smooth curve, is universally proportional to the length of the reference curve, times the wavenumber, independent of the geometry. We found an upper bound for the nodal intersections variance, depending on the arithmetic properties of the straight line. The considerations made establish a close relation between this problem and the theory of lattice points on spheres.

1. Introduction

1.1. Nodal intersections and lattice points on spheres

On the three-dimensional flat torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$ consider a real-valued eigenfunction of the Laplacian $F : \mathbb{T}^3 \to \mathbb{R}$, with eigenvalue λ^2:

$$(\Delta + \lambda^2)F = 0.$$

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The nodal set of F is the zero locus

$$\{x \in \mathbb{T}^3 : F(x) = 0\},$$

consisting of a union of smooth surfaces, possibly together with a set of lower dimension, i.e. curves and points (cf. [6,13]).

Let $\mathcal{C} \subset \mathbb{T}^3$ be a fixed straight line segment on the torus, of length L, parametrised by $\gamma(t) = t\alpha = t(\alpha_1, \alpha_2, \alpha_3)$, with $0 \le t \le L$, $\alpha \in \mathbb{R}^3$ and $|\alpha| = 1$.

We want to study the number of nodal intersections

$$\#\{x \in \mathbb{T}^3 : F(x) = 0\} \cap \mathcal{C}, \tag{1.1}$$

i.e., the number of zeros of F on $\mathcal{C}$, as $\lambda \to \infty$.

The Laplace eigenvalues ("energy levels") on $\mathbb{T}^3$ are $\lambda^2 = 4\pi^2 m$, where m is a natural number expressible as a sum of three integer squares. Let

$$\mathcal{E} = \mathcal{E}(m) := \{\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{Z}^3 : \mu_1^2 + \mu_2^2 + \mu_3^2 = m\} \tag{1.2}$$

be the set of all lattice points on the sphere of radius $\sqrt{m}$. Their cardinality equals $r_3(m)$, the number of ways that m can be written as a sum of three squares, and will be denoted

$$N = N_m := \#\mathcal{E} = r_3(m)$$

(see Section 4.1); it is also the dimension of the eigenspace relative to the eigenvalue $4\pi^2 m$. The eigenspace admits the L^2-orthonormal basis $\{e^{2\pi i \langle \mu, x\rangle}\}_{\mu \in \mathcal{E}}$, a general form of (complex-valued) eigenfunctions being

$$F(x) = \sum_{\mu \in \mathcal{E}} c_\mu e^{2\pi i \langle \mu, x\rangle},$$

with $c_\mu \in \mathbb{C}$ Fourier coefficients. We will henceforth consider only real-valued eigenfunctions.

1.2. Arithmetic random waves

One cannot expect to have any deterministic lower or upper bounds for the number of nodal intersections (1.1). Indeed, [15, Examples 1.1, 1.2] gives sequences of eigenfunctions F and curves $\mathcal{C}$ where $\mathcal{C}$ is contained in the nodal set for arbitrarily high energy, and planar curves with no nodal intersections at all, m arbitrarily large. Let us then consider the *random* Gaussian toral eigenfunctions ('arithmetic random waves' [12,13,10])

$$F(x) = \frac{1}{\sqrt{N}} \sum_{(\mu_1, \mu_2, \mu_3) \in \mathcal{E}} a_\mu e^{2\pi i \langle \mu, x\rangle} \tag{1.3}$$