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Discrepancy densities for planar and hyperbolic zero packing

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A R T I C L E I N F O

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A B S T R A C T

We study the problem of *geometric zero packing*, recently introduced by Hedenmalm [7]. There are two natural densities associated with this problem: the *discrepancy density* $\rho_{\mathbb{H}}$, given by

$$\rho_{\mathbb{H}} = \liminf_{r\to 1^-} \inf_f \frac{\int_{\mathbb{D}(0,r)} \left((1-|z|^2)|f(z)|-1\right)^2 \frac{dA(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)} \frac{dA(z)}{1-|z|^2}}$$

which measures the discrepancy in optimal approximation of $(1-|z|^2)^{-1}$ with the modulus of polynomials f, and its relative, the *tight discrepancy density* $\rho^*_{\mathbb{H}}$, which will trivially satisfy $\rho_{\mathbb{H}} \leq \rho^*_{\mathbb{H}}$. These densities have deep connections to the boundary behaviour of conformal mappings with k-quasiconformal extensions, which can be seen from Hedenmalm's result that the universal asymptotic variance Σ^2 is related to $\rho^*_{\mathbb{H}}$ by $\Sigma^2 = 1 - \rho^*_{\mathbb{H}}$. Here we prove that in fact $\rho_{\mathbb{H}} = \rho^*_{\mathbb{H}}$, resolving a conjecture by Hedenmalm in the positive. The natural planar analogues $\rho_{\mathbb{C}}$ and $\rho^*_{\mathbb{C}}$ to these densities make contact with work of Abrikosov on Bose–Einstein condensates. As a second result we prove that also $\rho_{\mathbb{C}} = \rho^*_{\mathbb{C}}$. The methods are based on Ameur, Hedenmalm and Makarov's Hörmander-type $\bar{\partial}$-estimates with polynomial growth control [2]. As a consequence we obtain

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sufficiency results on the degrees of approximately optimal polynomials.

1. Introduction

1.1. Hyperbolic discrepancy densities

Let $0 < r < 1$ and let f be a holomorphic function defined on the unit disk $\mathbb{D}$. We shall be concerned with the *hyperbolic discrepancy function* $\Phi_f(z, r)$, defined by

$$\Phi_f(z,r) = \big((1-|z|^2)|f(z)| - 1_{\mathbb{D}(0,r)}(z)\big)^2, \qquad z \in \mathbb{D}.$$

The intuition is that Φ_f measures the discrepancy between f and the hyperbolic metric $\vartheta(z) = (1 - |z|^2)^{-1}$. Since f is holomorphic, $\Delta \log|f(z)|$ is, considered as a distribution, a sum of point masses, while $\Delta \log \vartheta(z)$ is a smooth positive density. This constitutes a clear obstruction to obtain a perfect approximation with holomorphic f. The term *zero packing*, introduced by Hedenmalm [7], comes from the realization that this problem can be phrased in terms of optimally discretizing the smooth positive mass $\Delta \log \vartheta$ as a sum of point masses – corresponding to the zeros of the holomorphic function f.

Our main interest lies in the *hyperbolic discrepancy density* $\rho_{\mathbb{H}}$, and in a related object called the *tight hyperbolic discrepancy density* $\rho^*_{\mathbb{H}}$. Without further delay we proceed to define these. For polynomials f we consider the functionals

$$\rho_{\mathbb{H},r}(f) = \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z,r) \frac{dA(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)} \frac{dA(z)}{1-|z|^2}} = \frac{\int_{\mathbb{D}(0,r)} \Phi_f(z,r) \frac{dA(z)}{1-|z|^2}}{\log \frac{1}{1-r^2}}$$

and

$$\rho^*_{\mathbb{H},r}(f) = \frac{\int_{\mathbb{D}} \Phi_f(z,r) \frac{dA(z)}{1-|z|^2}}{\int_{\mathbb{D}(0,r)} \frac{dA(z)}{1-|z|^2}} = \frac{\int_{\mathbb{D}} \Phi_f(z,r) \frac{dA(z)}{1-|z|^2}}{\log \frac{1}{1-r^2}}.$$

In terms of these, the two densities are obtained as

$$\rho_{\mathbb{H}} = \liminf_{r \to 1^-} \inf_f \rho_{\mathbb{H},r}(f), \tag{1.1}$$

and

$$\rho^*_{\mathbb{H}} = \liminf_{r \to 1^-} \inf_f \rho^*_{\mathbb{H},r}(f), \tag{1.2}$$

where in both cases the infimum is taken over the set of all polynomials $\mathrm{Pol}(\mathbb{C})$.