Journal of Functional Analysis 272 (2017) 2933–2955

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Hyperbolic development and inversion of signature

Terry J. Lyons [a], Weijun Xu [b,*]

[a] Mathematical and Oxford-Man Institutes, University of Oxford, Woodstock road, Oxford, OX2 6GG, UK
[b] Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

ARTICLE INFO

Article history:
Received 2 July 2015
Accepted 22 December 2016
Available online 28 December 2016
Communicated by L. Gross

Keywords:
Signature
Hyperbolic development

ABSTRACT

We develop a simple procedure that allows one to explicitly reconstruct any piecewise linear path from its signature. The construction is based on the development of the path onto the hyperbolic space.

© 2016 Published by Elsevier Inc.

1. Introduction

A (Euclidean) path γ is a continuous function mapping some finite interval $[0, T]$ into $\mathbb{R}^d$. The length of γ is defined as

$$\|\gamma\| := \sup_{\mathcal{P} \subset [0,T]} \sum_{u_j \in \mathcal{P}} |\gamma_{u_{j+1}} - \gamma_{u_j}|,$$

where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[0, T]$, and

$$|\gamma_u| := \left(\sum_{j=1}^{d} |\gamma_u^{(j)}|^2 \right)^{\frac{1}{2}}$$

* Corresponding author.
E-mail addresses: tlyons@maths.ox.ac.uk (T.J. Lyons), weijun.xu@warwick.ac.uk (W. Xu).

http://dx.doi.org/10.1016/j.jfa.2016.12.024
0022-1236/© 2016 Published by Elsevier Inc.

is the Euclidean norm of the vector $\gamma_u = (\gamma_u^{(1)}, \ldots, \gamma_u^{(d)})^T$. We say γ has *bounded variation* if $\|\gamma\| < +\infty$.

There are two natural operations on the space of bounded variation paths: concatenation and inverse. For $\alpha : [0, S] \to \mathbb{R}^d$ and $\beta : [0, T] \to \mathbb{R}^d$, their concatenation $\alpha * \beta$ is defined as

$$\alpha * \beta(u) := \begin{cases} \alpha(u), u \in [0, S] \\ \beta(u - S) + \alpha(S) - \alpha(0), u \in [S, S + T] \end{cases}. \tag{1.1}$$

The inverse of a path $\gamma : [0, T] \to \mathbb{R}^d$ is defined by $\gamma^{-1}(u) := \gamma(T - u)$.

We say a path γ is irreducible if for every $s < t$, there exists no $u \in (s, t)$ such that $\gamma|_{[s,u]} = (\gamma|_{[u,t]})^{-1}$, where $\gamma|_{[s,u]}$ denotes the segment of γ restricted to the time interval $[s, u]$, and similarly for $(\gamma|_{[u,t]})^{-1}$.

If γ has bounded variation, then its derivative $\theta(t) = \dot{\gamma}(t)$ exists almost everywhere. We can re-parametrize γ in the fixed time interval $[0, 1]$ in such a way that

$$|\theta(t)| := |\dot{\gamma}(t)| \equiv L \tag{1.2}$$

for almost every $t \in [0, 1]$, where $|\cdot|$ is the Euclidean norm, and L is the length of γ. We call such a parametrization the *natural parametrization* of γ. Note that if $\gamma \in \mathcal{C}^1$ (at natural parametrization), then it is automatically irreducible.

Remark 1.1. The notion of natural parametrization defined in (1.2) is slightly different from the standard one in literature, as we parametrize the path in the unit interval $[0, 1]$ rather than $[0, L]$. As a consequence, γ has constant speed L instead of 1. We will see later that it will be convenient for us if we fix the time interval to be $[0, 1]$ instead of changing it with length.

For every path of bounded variation, one can associate to it a formal power series whose coefficients are iterated integrals of the path. This formal series is called the *signature* of the path, first introduced by K.T. Chen ([2,3]). Before we give the precise definition of signature, we first introduce a few notations.

We denote by $\{e_1, \ldots, e_d\}$ the standard basis of $\mathbb{R}^d$. For $n \geq 0$, a word w of length n is a sequence of n basis elements from the set $\{e_1, \ldots, e_d\}$ (with repetition allowed), and we use $|w|$ to denote the length of w. For simplicity, we will often write words as sequence of elements from the set $\{1, \ldots, d\}$. For example, $w = (2, 3, 1, 1)$ denotes the word (e_2, e_3, e_1, e_1), and $|w| = 4$. We also use $\emptyset$ to denote the empty word, which is the unique word with length 0. Given a word $w = (i_1, \ldots, i_n)$, we let

$$\mathbf{e}_w = e_{i_1} \otimes \cdots \otimes e_{i_n}.$$

With these notations, we now give a precise definition of the signature.

Download English Version:

https://daneshyari.com/en/article/5772232

Download Persian Version:

https://daneshyari.com/article/5772232

Daneshyari.com