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A monotonicity formula for mean curvature flow with surgery

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ABSTRACT

We prove a monotonicity formula for mean curvature flow with surgery. This formula differs from Huisken's monotonicity formula by an extra term involving the mean curvature. As a consequence, we show that a surgically modified flow which is sufficiently close to a smooth flow in the sense of geometric measure theory is, in fact, free of surgeries. This result is used in the analysis of mean curvature flow with surgery in Riemannian three-manifolds (cf. [5]).

1. Introduction

One of the main tools in the study of mean curvature flow is Huisken's monotonicity formula (cf. [12]). In the special case of smooth solution of mean curvature flow in $\mathbb{R}^3$, the monotonicity formula implies that the Gaussian integral

$$\int_{M_t} \frac{1}{4\pi(t_0 - t)}\, e^{-\frac{|x-p|^2}{4(t_0-t)}}$$

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is monotone decreasing in t, as long as $t_0 - t > 0$. This monotonicity property plays a crucial role in the analysis of singularities; see e.g. [6,10,16–18].

Our goal in this paper is to adapt the monotonicity formula to surgically modified flows. Motivated by earlier work of Hamilton [8,9] and Perelman [14,15] on the Ricci flow, Huisken and Sinestrari [13] introduced a notion of mean curvature flow with surgery for two-convex hypersurfaces in $\mathbb{R}^{n+1}$, where $n \geq 3$. In a joint work with Gerhard Huisken [4], we extended this construction to the case $n = 2$. As a result, we obtained a notion of mean curvature flow with surgery for embedded, mean convex surfaces in $\mathbb{R}^3$. One of the key ingredients in the proof is a sharp estimate for the inscribed radius established earlier in [2]. An alternative construction was given by Haslhofer and Kleiner [11]. We note that the surgery construction in [4] can be extended to flows of embedded, mean convex surfaces in three-manifolds; see [3] and [5] for details.

Throughout this paper, we will focus on mean curvature flow with surgery for embedded, mean convex surfaces in three-manifolds. We were unable to show that the Gaussian density considered by Huisken is monotone across surgery times. The reason is that, as we replace a neck by a cap, the area element may increase. To get around this technical problem, we modify the Gaussian density by including a term involving the mean curvature. More precisely, if M_t is a mean curvature flow with surgery in $\mathbb{R}^3$, we show that the quantity

$$\int_{M_t} \frac{1}{4\pi(t_0 - t)}\, e^{-\frac{|x-p|^2}{4(t_0-t)} - \frac{H}{200\, H_1}}$$

is monotone decreasing in t, as long as $t_0 - t \geq \frac{5}{9}\, H_1^{-2}$. Here, H_1 is a positive constant which represents the so-called surgery scale; in other words, each neck on which we perform surgery has radius between $\frac{1}{2H_1}$ and $\frac{1}{H_1}$. A similar monotonicity property holds for mean curvature flow with surgery in a Riemannian three-manifold.

In Section 2, we establish a number of auxiliary results. These results will be used in Section 3 to deduce a monotonicity formula for mean curvature flow with surgery in $\mathbb{R}^3$. In Section 4, we extend the monotonicity formula to solutions of mean curvature flow with surgery in Riemannian three-manifolds.

2. Behavior of a Gaussian integral under a single surgery

In this section, we analyze how the Gaussian integral changes under a single surgery. The strategy will be to estimate the integral of the Gaussian density over each vertical cross section. To that end, we require some preliminary estimates involving integrals over curves in $\mathbb{R}^2$. In the following, S^1 will denote the unit circle in $\mathbb{R}^2$ centered at the origin.

Lemma 2.1. *There exists a real number $\beta > 0$ with the following significance. Suppose that Γ is a curve in $\mathbb{R}^2$ which is β-close to the unit circle S^1 in the C^1-norm. Moreover, suppose that ψ is a real-valued function defined on Γ satisfying* $\sup_\Gamma |\psi - 1| < \beta$. *Then*