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Dual Kadec–Klee property and fixed points

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ABSTRACT

A Banach space X is said to have the dual Kadec–Klee property iff on the unit sphere of the dual space X^* the weak*-topology coincides with the norm-topology. A. Amini-Harandi and M. Fakhar extended a theorem of B. Ricceri concerning the Kadec–Klee property by showing that if X has the dual Kadec–Klee property then for every compact mapping $\varphi : B_{X^*} \to X \setminus \{0\}$ there exists some f in the unit sphere of X^* such that $\langle f, \varphi(f)\rangle = \|\varphi(f)\|$ and conjectured that this yields a necessary and sufficient condition for X to have the dual Kadec–Klee property. We prove here that it is almost the case: we introduce a weakening of the dual Kadec–Klee property, here called the NAKK* property, and show that this property is equivalent to the above property about compact mappings from B_{X^*} to $X \setminus \{0\}$.

1. Introduction

Recall that a Banach space X is said to have the Kadec–Klee property if the weak topology and the norm-topology of X agree on the unit sphere S_X of X. This property, which is weaker than being locally uniformly rotund, was introduced by M. Kadec in his

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proof that every separable Banach space can be equipped with an equivalent l.u.r. norm. Recall also that a function $f : X \to Y$ is said to be compact if it is continuous and its range $f(X)$ is relatively compact in Y.

In a recent paper [5], B. Ricceri was interested in the following problem of fixed point. Let X be an infinite-dimensional reflexive Banach space and $\varphi : S_X \to X^* \setminus \{0\}$ a continuous function; is there some $x \in S_X$ such that $\langle \varphi(x), x\rangle = \|\varphi(x)\|$? He proved that there are counterexamples in every infinite-dimensional Banach space if φ is not assumed to be compact. Nevertheless he has shown the following:

Theorem. *Let X be a reflexive infinite dimensional Banach space having the Kadec–Klee property. If $\varphi : S_X \to X^* \setminus \{0\}$ is a compact function then there exists some $\hat{x} \in S_X$ such that $\langle \varphi(\hat{x}), \hat{x}\rangle = \|\varphi(\hat{x})\|$.*

In fact since there exists a continuous retraction from the unit ball B_X onto S_X, this statement is equivalent to the following one where S_X is replaced by B_X.

Theorem. *Let X be a reflexive infinite dimensional Banach space having the Kadec–Klee property. If $\varphi : B_X \to X^* \setminus \{0\}$ is a compact function then there exists some $\hat{x} \in S_X$ such that $\langle \varphi(\hat{x}), \hat{x}\rangle = \|\varphi(\hat{x})\|$.*

In the same paper B. Ricceri asked whether this property could characterize the Kadec–Klee property among the reflexive Banach spaces, and in [6] the present author proved that it is the case.

Recently A. Amini-Harandi and M. Fakhar (in the so far unpublished paper [1]) considered a variant of the previous problem. A Banach space X is said to have the dual Kadec–Klee property if the weak*-topology and the norm-topology of X^* agree on the unit sphere S_{X^*} of X^*. When X is separable this is equivalent to the following property: *If the sequence (x_n^*) weak*-converges to x^* in X^* and $\|x_n^*\| \to \|x^*\|$ then $\|x_n^* - x^*\| \to 0$.*

Following the scheme of proof of Ricceri in [5] these two authors prove in [1] the following

Theorem. *Let X be an infinite-dimensional Banach space having the dual Kadec–Klee property. If $\varphi : S_{X^*} \to X \setminus \{0\}$ is a compact function then there exists some $\hat{x}^* \in S_{X^*}$ such that $\langle \hat{x}^*, \varphi(\hat{x}^*)\rangle = \|\varphi(\hat{x}^*)\|$.*

The authors remark that this statement extends the theorem proved by Ricceri in [5]: indeed if X is a reflexive Banach space then it has the Kadec–Klee property iff X^* has the dual Kadec–Klee property.

And in the same paper they conjectured that for an infinite-dimensional Banach space this property characterizes the dual Kadec–Klee property. In fact as we shall see below, one can introduce a slightly weaker property that we call here the norm-attaining dual Kadec–Klee property (NAKK* for short) and show in the same way that in the previous