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# On the classification of positions and complex structures in Banach spaces

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ABSTRACT

A topological setting is defined to study the complexities of the relation of equivalence of embeddings (or "position") of a Banach space into another and of the relation of isomorphism of complex structures on a real Banach space. The following results are obtained: a) if $X$ is not uniformly finitely extensible, then there exists a space $Y$ for which the relation of position of $Y$ inside $X$ reduces the relation $E_0$ and therefore is not smooth; b) the relation of position of $\ell_p$ inside $\ell_p$, or inside $L_p$, $p \neq 2$, reduces the relation $E_1$ and therefore is not reducible to an orbit relation induced by the action of a Polish group; c) the relation of position of a space inside another can attain the maximum complexity $E_{\max}$; d) there exists a subspace of $L_p$, $1 \leq p < 2$, on which isomorphism between complex structures reduces $E_1$

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and therefore is not reducible to an orbit relation induced by the action of a Polish group.


## 1. Introduction

In this paper we are interested in defining a topological setting to compute the complexity of certain natural equivalence relations appearing in the theory of positions and/or complex structures. Our objective is to provide examples towards the idea that these relations are not "well classifiable", or in other words, to obtain high lower bounds of complexity for natural instance of these relations. Our starting points are the previous results in which a continuum of equivalence classes were already obtained, without information on the complexity of the associated equivalence relation: examples of spaces with a continuum of mutually non-isomorphic complex structures [1], or examples of classical spaces with continuum many different positions inside another, see [6] and [18].

In this introduction we recall some basics of the theories of positions of Banach spaces, of complex structures, as well as of classification of analytic equivalence relations on Polish spaces. In section 2, after defining the appropriate topological setting, we obtain lower bounds for the complexity of position of a space inside another, in different cases. We prove that if $X$ is not uniformly finitely extensible, then there exists a space $Y$ for which the relation of position of $Y$ inside $X$ reduces the relation $E_0$ and therefore is not smooth (Theorem 2.7). Through a result about complexity of positions inside $\ell_p$-sums of non-uniformly extensible spaces (Proposition 2.10), we extend this and prove that the relation of position of $\ell_p$ inside $\ell_p$, or inside $L_p$, $p \neq 2$, reduces the relation $E_1$ and therefore is not reducible to an orbit relation induced by the action of a Polish group, Theorem 2.12. Then through the study of complemented positions we use the main result of [10] to show that the complexity of positions may be $E_{\max}$, the maximum complexity of analytic equivalence relations, Proposition 2.15. We end the section by providing the appropriate topological setting to study complex structures. In section 3, we describe an example to prove that there exists a subspace of $L_p$, $1 \leq p < 2$, on which isomorphism between complex structures reduces $E_1$ and therefore is not reducible to an orbit relation induced by the action of a Polish group.

### 1.1. Positions of Banach spaces

The notion of relative positions of Banach spaces arose in [5] where the definition of *automorphic* space was first introduced in connection with a classical result of Lindenstrauss and Rosenthal [15]: $c_0$ *has the property that every isomorphism between two of its infinite codimensional subspaces can be extended to an automorphism of the whole space.*