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Compact double differences of composition operators on the Bergman spaces ☆

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As is well known on the weighted Bergman spaces over the unit disk, compactness of differences of two composition operators is characterized by certain cancellation property of the inducing maps at every "bad" boundary point, which makes each composition operator in the difference fail to be compact. Recently, the second and third authors obtained a result implying that double difference cancellation is not possible for linear combinations of three composition operators. In this paper, we obtain a complete characterization for compact double differences formed by four composition operators. Applying our characterization, we easily recover known results on linear combinations of two or three composition operators. As another application, we also show that double difference cancellation is possible for linear combinations of four composition operators by constructing an explicit example of a compact double difference formed by two noncompact differences. In spite of such an example, our characterization also shows that double difference cancellation may occur in the global sense only, and that genuine double difference cancellation is not possible in a certain local sense.

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1. Introduction

Let $\mathcal{S} = \mathcal{S}(\mathbf{D})$ be the class of all holomorphic self-maps of the unit disk $\mathbf{D}$ of the complex plane $\mathbf{C}$. Each $\varphi \in \mathcal{S}$ induces a composition operator $C_\varphi : H(\mathbf{D}) \to H(\mathbf{D})$ defined by

$$C_\varphi f := f \circ \varphi,$$

where $H(\mathbf{D})$ is the class of all holomorphic functions on $\mathbf{D}$. An extensive study on the theory of composition operators has been established during the past four decades on various settings. We refer to standard references [9] and [27] for various aspects on the theory of composition operators acting on holomorphic function spaces.

We first recall our function spaces to work on. Let dA be the area measure on $\mathbf{D}$ normalized to have total mass 1. For $\alpha > -1$, put

$$dA_\alpha(z) := (\alpha + 1)(1 - |z|^2)^\alpha dA(z), \qquad z \in \mathbf{D};$$

the constant $\alpha + 1$ is chosen so that $A_\alpha(\mathbf{D}) = 1$. Now, for $0 < p < \infty$, the α-weighted Bergman space $A^p_\alpha(\mathbf{D})$ is the space of all $f \in H(\mathbf{D})$ such that the "norm"

$$\|f\|_{A^p_\alpha} := \left\{ \int_{\mathbf{D}} |f|^p \, dA_\alpha \right\}^{1/p}$$

is finite. As is well-known, the space $A^p_\alpha(\mathbf{D})$ equipped with the norm above is a Banach space for $1 \le p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f, g) \mapsto \|f - g\|^p_{A^p_\alpha}$.

As is well known in the setting of $\mathbf{D}$, every composition operator is bounded on well-known function spaces such as the weighted Bergman spaces $A^p_\alpha(\mathbf{D})$ and the Hardy spaces $H^p(\mathbf{D})$ due to the Littlewood Subordination Principle; see [9] or [27] for precise definition of the Hardy spaces $H^p(\mathbf{D})$. So, boundedness on those spaces is out of question and much efforts have been expended in the early stage on characterizing those maps in $\mathcal{S}$ which induce compact composition operators. An early result of Shapiro and Taylor [29] in 1973 showed that the condition

$$\lim_{|z| \to 1} \frac{1 - |z|}{1 - |\varphi(z)|} = 0 \tag{1.1}$$

is necessary for $\varphi \in \mathcal{S}$ to induce a compact composition operator on $H^2(\mathbf{D})$ (and hence for the general Hardy spaces). This means (see Section 2.4) that the non-existence of the angular derivative of the inducing map at any boundary point is a necessary condition for the compactness of a composition operator on the Hardy spaces. However, (1.1) turned out to be not sufficient. In fact, later in 1989 Shapiro [26] completely characterized the compactness of composition operators on the Hardy spaces by finding the precise formula