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Hitting times, functional inequalities, Lyapunov conditions and uniform ergodicity

Patrick Cattiaux [a], Arnaud Guillin [b,c,*]

[a] *Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université Paul Sabatier, 118 route de Narbonne, F-31062 Toulouse cedex 09, France*
[b] *Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal, avenue des Landais, F-63177 Aubière, France*
[c] *Institut Universitaire de France, France*

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A B S T R A C T

The use of Lyapunov conditions for proving functional inequalities was initiated in [5]. It was shown in [4,30] that there is an equivalence between a Poincaré inequality, the existence of some Lyapunov function and the exponential integrability of hitting times. In the present paper, we close the scheme of the interplay between Lyapunov conditions and functional inequalities by

- showing that strong functional inequalities are equivalent to Lyapunov type conditions;
- showing that these Lyapunov conditions are characterized by the finiteness of generalized exponential moments of hitting times.

We also give some complement concerning the link between Lyapunov conditions and integrability property of the invariant probability measure and as such transportation inequalities, and we show that some “unbounded Lyapunov condi-

* Corresponding author.
E-mail addresses: cattiaux@math.univ-toulouse.fr (P. Cattiaux), guillin@math.univ-bpclermont.fr (A. Guillin).

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tions" can lead to uniform ergodicity, and coming down from infinity property.

1. Introduction

Let D be some smooth open domain in $\mathbb{R}^d$. In this paper, we will mainly consider the differential operator defined for smooth functions $f \in C^\infty(D)$ by

$$Lf = \frac{1}{2}\sum_{ij}(\sigma\,\sigma^*)_{ij}(x)\,\frac{\partial^2_{ij} f}{\partial x_{ij}} + \sum_i b_i(x)\frac{\partial_i f}{\partial x_i}\,,$$

where σ is an $\mathbb{R}^{d\times m}$ smooth and bounded (for simplicity $C_b^\infty(\bar{D})$) matrix field and b a $C^\infty(\bar{D})$ vector field.

We may see L as the infinitesimal generator of a diffusion process associated to the stochastic differential equation (SDE)

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \quad , \quad X_0 = x\,,$$

where B_t is a usual $\mathbb{R}^m$-Brownian motions when $D = \mathbb{R}^d$, or to the reflected SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + dN_t, \quad \int_0^t \mathbf{1}_{\partial D}(X_s)dN_s = N_t \quad X_0 = x\,,$$

if D is some smooth subdomain.

The domain $\mathcal{D}(L)$ of L (viewed as a generator) is thus some extension of the set of smooth and compactly supported functions $C_c^\infty(\bar{D})$ such that the normal derivative $\frac{\partial f}{\partial n}$ vanishes on ∂D (if ∂D is nonvoid). This corresponds to normal reflection or to a Neumann condition on the boundary. We also define P_t the associated semi-group

$$P_t f(x) = \mathbb{E}_x(f(X_t))$$

which is defined for bounded functions f.

In order to use classical results in PDE theory we will also assume that L is uniformly elliptic, i.e.

$$\sigma\,\sigma^* \,\geq\, a\,Id$$

in the sense of quadratic forms for some $a > 0$, or more generally that L is uniformly strongly hypo-elliptic in the sense of Bony (see [15]) and that the boundary is noncharacteristic. For details we refer to [17].