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Approximation in Morrey spaces ☆

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A new subspace of Morrey spaces whose elements can be approximated by infinitely differentiable compactly supported functions is introduced. Consequently, we give an explicit description of the closure of the set of such functions in Morrey spaces. A generalization of known embeddings of Morrey spaces into weighted Lebesgue spaces is also obtained.

1. Introduction

Morrey spaces are widely used in applications to regularity properties of solutions to PDE including the study of Navier–Stokes equations (see [17] and references therein).

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Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non-separable and that the usual classes of nice functions are not dense in such spaces.

The theory of Morrey spaces goes back to Morrey [8] who considered related integral inequalities in connection with regularity properties of solutions to nonlinear elliptic equations. In the form of Banach spaces of functions, called thereafter Morrey spaces, the ideas of Morrey [8] were further developed by Campanato [4]. A more systematic study of these (and even more general) spaces was carried out by Peetre [9] and Brudnyi [3]. We refer to the books [1,6,10,15,17] and the overview [11] for additional references and basic properties and generalizations of Morrey spaces. We also refer to [2] for Harmonic Analysis in Morrey spaces and the special issue with Editorial [14] for a discussion on related function spaces.

In [20] it was observed that the set of functions in Morrey spaces for which the translation is continuous in Morrey norm plays an important role in approximation. This was sketched in [20, Proposition 3] and also discussed in [5] and [7]. A description of this set of functions in easily verified terms seems to be a difficult task for unbounded domains; at least the authors were unable to find any one in the literature. In [20] and [7] there were given some results on equivalence between belonging to Zorko space and approximation by mollifiers.

In this paper we introduce some subspaces of Morrey spaces by adding some "vanishing" type conditions. One of them is similar to the already known vanishing property, but related to the behavior at infinity instead of that at the origin. Another is connected with the truncation of functions to the exterior of large balls. These two additional properties, together with the vanishing property at the origin, allow us to show that all elements in this new subspace, denoted in the sequel by $V_{0,\infty}^{(*)}L^{p,\lambda}$, may be approximated by C_0^∞ functions in Morrey norm. In particular, dilation type identity approximations with integrable kernels strongly converges for all the functions in $V_{0,\infty}^{(*)}L^{p,\lambda}$.

As we shall see, the set $V_{0,\infty}^{(*)}L^{p,\lambda}$ is strictly smaller than Zorko class. An example of a Morrey function belonging to Zorko class but not belonging to this new subspace is given below. Moreover, $V_{0,\infty}^{(*)}L^{p,\lambda}$ is closed in the Morrey space. Consequently, we give an explicit description of the closure of C_0^∞ in Morrey spaces. This closure plays an important role in Harmonic Analysis on Morrey spaces, including Calderón–Zygmund theory, since its dual constitutes a predual of Morrey spaces (cf. [2]). Preduals of Morrey spaces have been studied by many authors, see the recent book [1] and the paper [12] for further details and references.

In addition to consideration of smaller subspaces of Morrey spaces, we also generalize some known embeddings of these spaces into weighted Lebesgue spaces. Various examples are presented showing the difference between all those spaces.

The paper is organized as follows. After some notation and preliminaries on Morrey spaces, the main ideas and results are given in Sections 3, 4, 5 and 6. In Section 3 we introduce new Morrey subspaces and discuss some of their properties, including the invariance with respect to convolutions with integrable kernels. The relation between