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On Schatten-class perturbations of Toeplitz operators

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ABSTRACT

Let D denote the unit ball or the unit polydisc in $\mathbb{C}^n$ with $n \geq 2$. For $1 \leq p \leq 2n$ in the case of the ball and $1 \leq p < \infty$ for the polydisc, we show that a bounded operator S on the Hardy space $H^2(D)$ commutes with all analytic Toeplitz operators modulo the Schatten class $\mathcal{S}_p$ if and only if $S = X + K$ with an analytic Toeplitz operator X and an operator $K \in \mathcal{S}_p$. This partially answers a question of Guo and Wang [14]. For $1 \leq p < \infty$ and a strictly pseudoconvex or bounded symmetric and circled domain $D \subset \mathbb{C}^n$, we show that a given operator S on $H^2(D)$ is a Schatten-p-class perturbation of a Toeplitz operator if and only if $T_\theta^* S T_\theta - S \in \mathcal{S}_p$ for every inner function θ on D.

1. Introduction and main results

Let $D \subset \mathbb{C}^n$ be either the unit ball $\mathbb{B}_n$ or the unit polydisc $\mathbb{D}^n$. In [14] Guo and Wang asked for a characterization of those bounded operators S on the Hardy space $H^2(D)$ that commute with all analytic Toeplitz operators modulo the Schatten classes

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$$\mathcal{S}_p = \{X \in B(H^2(D)) : \|X\|_p = (\mathrm{tr}|X|^p)^{1/p} < \infty\}.$$

Note that there is a chain of inclusions

$$\mathcal{S}_0 \subset \mathcal{S}_p \subset \mathcal{S}_q \subset \mathcal{S}_\infty \qquad (1 \leq p < q < \infty),$$

where $\mathcal{S}_0$ stands for the finite-rank and $\mathcal{S}_\infty$ for the compact operators on $H^2(D)$. Questions concerning the p-essential commutant

$$\mathcal{T}_a^{\mathrm{ec},p} = \{S \in B(H^2(D)) : [S, X] = SX - XS \in \mathcal{S}_p \text{ for every } X \in \mathcal{T}_a\}$$

of the set of all analytic Toeplitz operators $\mathcal{T}_a = \{T_f : f \in H^\infty(D)\}$ date back at least to the work of Douglas and Sarason on Toeplitz operators on the unit disc. A complete characterization of the essential commutant $\mathcal{T}_a^{ec} = \mathcal{T}_a^{ec,\infty}$ of the analytic Toeplitz operators on the unit disc was given by Davidson [4] in 1977. A generalization of Davidson's result to the unit ball was obtained in 1997 by Ding and Sun [9] (see also Le [17] for related work). In a recent paper of Everard and the first two authors [8] Davidson's result is extended to the case of smoothly bounded strictly pseudoconvex domains $D \subset \mathbb{C}^n$. More precisely, it is shown that

$$\mathcal{T}_a^{\mathrm{ec}} = \{T_f + K : K \text{ compact}, \ f \in L^\infty(\sigma) \text{ with } H_f \text{ compact}\},$$

where σ denotes the normalized surface measure on ∂D and H_f is the Hankel operator with symbol f. On the unit disc, a well known theorem of Hartman says that H_f is compact if and only if $f \in H^\infty + C$ and thus we recover the main result of [4].

At the other end of the chain, namely for $p = 0$, Guo and Wang completely solved both the ball and the polydisc case extending corresponding one-dimensional results of Gu [13]. They proved in [14] that

$$\mathcal{T}_a^{\mathrm{ec},0} = \mathcal{T}_a + \mathcal{S}_0 \qquad (\text{if } D = \mathbb{B}_n \text{ or } D = \mathbb{D}^n \text{ and } n \geq 2)$$

while, for $n = 1$, $\mathcal{T}_a$ has to be replaced by the set of all Toeplitz operators T_f where f is the sum of a rational function and a bounded analytic function.

One of the goals of the present paper is to fill in at least some of the gaps between $p = 0$ and $p = \infty$. Using results on the compactness and the Schatten-class membership of Hankel operators on $D = \mathbb{D}^n$ and $D = \mathbb{B}_n$ due to Cotlar and Sadoski [2] and Fang and Xia [12], respectively, we are able to prove the following result:

1. Theorem. *Let $n \geq 2$ be given. If either*

$$D = \mathbb{B}_n \text{ and } 1 \leq p \leq 2n \qquad \text{or} \qquad D = \mathbb{D}^n \text{ and } 1 \leq p < \infty,$$

then the identity