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Large gaps between consecutive prime numbers containing perfect k-th powers of prime numbers

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ABSTRACT

Let $k \geq 2$ be a fixed natural number. We establish the existence of infinitely many pairs of consecutive primes p_n, p_{n+1} satisfying

$$p_{n+1} - p_n \geq c \, \frac{\log p_n \, \log_2 p_n \, \log_4 p_n}{\log_3 p_n} ,$$

with c being a fixed positive constant, for which the interval (p_n, p_{n+1}) contains the k-th power of a prime number.

1. Introduction and statement of main theorem

In their paper [3], K. Ford, D.R. Heath-Brown and S. Konyagin prove the existence of infinitely many "prime-avoiding" perfect k-th powers for any positive integer k.

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They give the following definition of **prime avoidance**: an integer m is called prime avoiding with constant c if $m+u$ is composite for all integers u satisfying

$$|u| \leq c\, g_1(m)\,,$$

with c being a positive constant and

$$g_1(m) = \frac{\log m \log_2 m \log_4 m}{(\log_3 m)^2}\,.$$

Here $\log_k x := \log(\log_{k-1} x)$.

In [11] the authors of the present paper extended this result by proving the existence of infinitely many prime-avoiding k-th powers of prime numbers.

Their method of proof consists in a combination of the method of [3] with the matrix method of the first author [9]. The matrix $\mathcal{M}$ employed in this technique is of the form:

$$\mathcal{M} = (a_{r,u})_{\substack{1\leq r\leq P(x)^{D-1}\\ u\in\mathcal{B}}},$$

with $P(x)$ being a product of many small prime numbers and D is a fixed positive integer, where the *rows*

$$R(r) = \{a_{r,u}\ :\ u\in\mathcal{B}\}$$

of the matrix are *translates* – in closer or wider sense – of the *base-row* $\mathcal{B}$. Moreover, $\mathcal{B}$ is contained in an interval of consecutive integers and consists of the few integers that are coprime to $P(x)$.

The columns of the matrix $\mathcal{M}$ are arithmetic progressions (or – in the case of [11] – shifted powers of elements of arithmetic progressions). The appearance of primes in these arithmetic progressions can be studied using results on primes in arithmetic progressions.

The construction of the base-row $\mathcal{B}$ in its simplest form has been carried out by Paul Erdős [2] and R.A. Rankin [15] in their papers on large gaps between consecutive primes. They obtain the following result: Infinitely often

$$p_{n+1} - p_n \geq c\, g_1(n)\,,$$

where $c > 0$ is a fixed constant. Until recently all improvements only concerned the constant c (cf. [10,13,16]).

In the papers [4,5,12] finally the function g_1 has been replaced by a function of a higher order of magnitude, solving a longstanding problem of Erdős.

K. Ford, B.J. Green, S. Konyagin, J. Maynard and T. Tao [5] have proved the following result:

$$p_{n+1} - p_n \geq c\, g_2(n)$$