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Extending positive definite functions from subgroups of nilpotent locally compact groups

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Continuing research begun in [15] and [16], we investigate the problem of when a closed subgroup H of a nilpotent locally compact group G has the property that every continuous positive definite function on H extends to some such function on G. The main result is an identification of all subgroups of the Heisenberg groups which share this property.

1. Introduction

Let G be a locally compact group and H a closed subgroup of G. Then H is called an *extending subgroup* of G if every continuous positive definite function on H extends to some continuous positive definite function on G. This is equivalent to that the restriction map $\phi \to \phi|_H$ between the Fourier–Stieltjes algebras $B(G)$ and $B(H)$ (compare [6]) is surjective [19]. We say that G has the *extension property* if each closed subgroup of G is

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extending. These properties have been studied by several authors (see [2,4,8–11,15,16,18] and [19]).

Let G be a locally compact group and H a closed subgroup of G. Then G is called a $[\mathrm{SIN}]_H$-group (a SIN-group) if G possesses a neighbourhood basis $\mathcal{V}$ of the identity such that, for all $V \in \mathcal{V}$, $h^{-1}Vh = V$ for all $h \in H$ ($x^{-1}Vx = V$ for all $x \in G$). It was independently shown by Cowling and Rodway [4] and by Henrichs [8] that a closed subgroup H of G is extending if G is an $[\mathrm{SIN}]_H$-group. Thus a closed subgroup which is either compact or central in G, is extending, and so are open subgroups. In particular, every SIN-group has the extension property. Conversely, a connected group G has the extension property only if it is a SIN-group (equivalently, G is the direct product of a vector group and a compact group) [4, Corollary 2]. This latter result can also directly be deduced from [10, Theorem 2]. In [15] it was shown that if G is a nilpotent locally compact group which has the extension property and is either compactly generated or has no non-trivial compact elements, then G must be a SIN-group. On the other hand, in [9] an example of a semi-direct product $\mathbb{R} \rtimes \mathbb{Z}$ was given which fails to be a SIN-group and nevertheless shares the extension property.

The problem of identifying the extending subgroups of a given group has turned out to be difficult, even for special classes of locally compact groups. This is, for instance, exemplified by a result due to Douady (Proposition 1.2) and also by results obtained in [15]. In this paper we pursue the investigation of [15] and [16]. Our most complete results concern simply connected 2-step nilpotent Lie groups. In Section 4 we determine the extending subgroups of any one of the Heisenberg groups H_n, $n \in \mathbb{N}$, and in Section 5 we characterize the connected abelian extending subgroups of the universal simply connected 2-step nilpotent Lie groups W_n, $n \geq 2$. Both of these results build on extensibility criteria for subgroups of connected nilpotent groups G, which are contained in the second member of the ascending central series of G (Section 3). Finally, in Section 6 we present some additional results, which might prove useful for further studies.

2. Preliminaries

We let $P(G)$ be the set of all continuous positive definite functions on G and $P^1(G) = \{\phi \in P(G) : \phi(e) = 1\}$. Moreover, $\mathrm{ex}(P^1(G))$ will denote the set of all extreme points of the convex set $P^1(G)$ (that is, those $\phi \in P^1(G)$ which cannot be written as a non-trivial convex combination of two distinct functions in $P^1(G)$). Then $\phi \in P^1(G)$ belongs to $\mathrm{ex}(P^1(G))$ if and only if π_ϕ, the cyclic unitary representation associated to ϕ through the GNS-construction, is irreducible. A character of G is a continuous homomorphism from G into the circle group $\mathbb{T}$. When G is abelian, $\mathrm{ex}(P^1(G))$ equals the dual group of G, the set of all characters of G. For basic properties of positive definite functions and unitary representations we refer to [5,12] and [17]. In particular, note that if $\phi \in P^1(G)$ and $y \in G$ is such that $|\phi(y)| = 1$, then $\phi(xy) = \phi(x)\phi(y)$ for all $x \in G$ [12, Theorem (32.4)(v)]. If π is a unitary representation of G and $\xi \in \mathcal{H}(\pi)$, the Hilbert