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Time-frequency analysis of Born–Jordan pseudodifferential operators

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A B S T R A C T

Born–Jordan operators are a class of pseudodifferential operators arising as a generalization of the quantization rule for polynomials on the phase space introduced by Born and Jordan in 1925. The weak definition of such operators involves the Born–Jordan distribution, first introduced by Cohen in 1966 as a member of the Cohen class. We perform a time-frequency analysis of the Cohen kernel of the Born–Jordan distribution, using modulation and Wiener amalgam spaces. We then provide sufficient and necessary conditions for Born–Jordan operators to be bounded on modulation spaces. We use modulation spaces as appropriate symbols classes.

1. Introduction

In 1925 Born and Jordan [3] introduced for the first time a rigorous mathematical explanation of the notion of "quantization". This rule was initially restricted to polynomials

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as symbol classes but was later extended to the class of tempered distribution $\mathcal{S}'(\mathbb{R}^{2d})$ [2,6]. Roughly speaking, a quantization is a rule which assigns an operator to a function (called symbol) on the phase space $\mathbb{R}^{2d}$. The Born–Jordan quantization was soon superseded by the most famous Weyl quantization rule proposed by Weyl in [34], giving rise to the well-known Weyl operators (transforms) (see, e.g. [35]).

Recently there has been a regain in interest in the Born–Jordan quantization, both in Quantum Physics and Time-frequency Analysis [16]. The second of us has proved that it is the correct rule if one wants matrix and wave mechanics to be equivalent to quantum theories [15]. Moreover, as a time-frequency representation, the Born–Jordan distribution has been proved to be better than the Wigner distribution since it damps very well the unwanted "ghost frequencies", as shown in [2,33]. For a throughout and rigorous mathematical explanation of these phenomena we refer to [13] whereas [23, Chapter 5] contains the relevant engineering literature about the geometry of interferences and kernel design.

To be more specific, the (cross-)Wigner distribution of signals f, g in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is defined by

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i y\omega} f(x+\frac{y}{2})\overline{g(x-\frac{y}{2})}\, dy. \tag{1}$$

The Weyl operator $\mathrm{Op_W}(a)$ with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ can be defined in terms of the Wigner distribution by the formula

$$\langle \mathrm{Op_W}(a) f, g\rangle = \langle a, W(g,f)\rangle.$$

For $z = (x,\omega)$, consider the Cohen kernel

$$\Theta(z) := \mathrm{sinc}(x\omega) = \begin{cases} \dfrac{\sin(\pi x\omega)}{\pi x\omega} & \text{for } \omega x \neq 0 \\ 1 & \text{for } \omega x = 0. \end{cases} \tag{2}$$

The (cross-)Born–Jordan distribution $Q(f,g)$ is then defined by

$$Q(f,g) = W(f,g) * \Theta_\sigma, \quad f,g \in \mathcal{S}(\mathbb{R}^d), \tag{3}$$

where Θ_σ is the symplectic Fourier transform recalled in (22) below. Likewise the Weyl operator, a Born–Jordan operator with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ can be defined as

$$\langle \mathrm{Op_{BJ}}(a) f, g\rangle = \langle a, Q(g,f)\rangle \quad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{4}$$

Any pseudodifferential operator admits a representation in the Born–Jordan form $\mathrm{Op_{BJ}}(a)$, as stated in [12].