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Iterations of the projection body operator and a remark on Petty's conjectured projection inequality

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ABSTRACT

We prove that if a convex body has an absolutely continuous surface area measure, whose density is sufficiently close to a constant function, then the sequence $\{\Pi^m K\}$ of convex bodies converges to the ball with respect to the Banach–Mazur distance, as $m \to \infty$. Here, Π denotes the projection body operator. Our result allows us to show that the ellipsoid is a local solution to the conjectured inequality of Petty and to improve a related inequality of Lutwak.

1. Introduction

For a convex body $K \subset \mathbb{R}^d$ (i.e. a convex, compact set with non-empty interior) the support function $h_K : \mathbb{S}^{d-1} \to \mathbb{R}_+$ is defined as $h_K(\xi) = \max\{\langle x, \xi\rangle : x \in K\}$ where we denote by $\langle x, y\rangle$ the inner product of two vectors $x, y \in \mathbb{R}^d$.

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The projection body of a convex body $K \subset \mathbb{R}^d$ is defined as the body with support function

$$h_{\Pi K}(x) = \left|K|x^\perp\right|, \text{ for all } x \in \mathbb{S}^{d-1},$$

where $K|x^\perp$ denotes the orthogonal projection of K onto the subspace $x^\perp = \{y \in \mathbb{R}^d : \langle x, y\rangle = 0\}$ and we write $|A|$ for the n-dimensional Lebesgue measure (volume) of a measurable set $A \subset \mathbb{R}^d$, where $n = 1, \dots, d$ is the dimension of the minimal flat containing A.

We note that a direct application of Cauchy's projection formula gives

$$h_{\Pi K}(x) = \frac{1}{2} \int\limits_{\mathbb{S}^{d-1}} |\langle x, y\rangle| dS_K(y), \; x \in \mathbb{S}^{d-1},$$

where S_K is the surface area measure of K, viewed as a measure on $\mathbb{S}^{d-1}$. When S_K is absolutely continuous (with respect to the Lebesgue measure on the sphere), its density f_K is the called curvature function of K (for more details see Section 8.3 in [7] or Chapter 4 in [3]).

The main goal of this paper is to study the iterations of the projection body operator. It is easy to see that $\Pi^2 K = 4K$, if $d = 2$ and K is symmetric (i.e. centrally symmetric with center at the origin, or $K = -K$). It is also easy to see that the projection body of the Euclidean ball B_2^d is again, up to a dilation, B_2^d. Moreover, the same property is true for a unit cube B_∞^d. Weil [18] proved that if K is a polytope in $\mathbb{R}^d$, then $\Pi^2 K$ is homothetic to K if and only if K is a linear image of Cartesian products of planar symmetric polygons or line segments. But no other description of fixed points of the projection body operator is known as well as not much is known about possible convergence of the sequence $\Pi^m K$.

We recall that the *Banach–Mazur distance* between symmetric bodies K and L is defined as

$$d_{BM}(K, L) = \inf\{t \geq 1 : L \subset TK \subset tL, \text{ for some } T \in \mathrm{GL}(d)\}.$$

The distance turns out to be extremely useful when one would like to study questions invariant under the linear transformations.

Clearly, the example of the cube and Weil's result tells us that one cannot expect in general that $\Pi^m K \to B_2^d$, with respect to the Banach–Mazur distance. It seems more plausible, however, that $\Pi^m K \to B_2^d$, for $d \geq 3$, if K has an absolutely continuous surface area measure. We refer to [3, Chapter 4, Problem 4.4] or [17, Section 10.9] for more information about this problem. We are going to show the following:

Theorem 1.1. *Let $d \geq 3$. There exists an $\varepsilon_d > 0$ such that for any convex body K in $\mathbb{R}^d$, with absolutely continuous surface area measure and the curvature function f_K*