Journal of Functional Analysis 272 (2017) 705–737

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

On algebras generated by Toeplitz operators and their representations ☆

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ARTICLE INFO

Article history:
Received 15 March 2016
Accepted 16 September 2016
Available online 21 September 2016
Communicated by D. Voiculescu

MSC:
primary 47B35, 47L80
secondary 32A36

Keywords:
Operator algebras
Weighted Bergman space
Semi-classical limit
Index

ABSTRACT

We study Banach and C^*-algebras generated by Toeplitz operators acting on weighted Bergman spaces $\mathcal{A}^2_\lambda(\mathbb{B}^2)$ over the complex unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. Our key point is an orthogonal decomposition of $\mathcal{A}^2_\lambda(\mathbb{B}^2)$ into a countable sum of infinite dimensional spaces, each one of which can be identified with a differently weighted Bergman space $\mathcal{A}^2_\mu(\mathbb{D})$ over the complex unit disk $\mathbb{D}$. Moreover, all elements of the above algebras leave each of the summands in the above decomposition invariant and their restriction to each level acts as a compact perturbation of a Toeplitz operator on $\mathcal{A}^2_\mu(\mathbb{D})$.

The symbols of the generating Toeplitz operators are chosen to be suitable extensions to $\mathbb{B}^2$ of families $\mathcal{S}$ of bounded functions on $\mathbb{D}$. Symbol classes $\mathcal{S}$ that generate important classical commutative and non-commutative Toeplitz algebras in $\mathcal{L}(\mathcal{A}^2_\mu(\mathbb{D}))$ are of particular interest. In this paper we discuss various examples. In the case of $\mathcal{S} = C(\overline{\mathbb{D}})$ and $\mathcal{S} = C(\overline{\mathbb{D}}) \otimes L_\infty(0,1)$ we characterize all irreducible representations

☆ This work was partially supported by CONACYT Project 238630, México and through the *Deutsche Forschungsgemeinschaft*, DFG Sachmittelbeihilfe BA 3793/4-1.

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http://dx.doi.org/10.1016/j.jfa.2016.09.013

of the resulting Toeplitz operator C^*-algebras. Their Calkin algebras are described and index formulas are provided.

1. Introduction

One of the common strategies in the study of Toeplitz operators T_a consists in selecting symbol subclasses S of L_∞ so that the properties of T_a with $a \in S$ and of the algebra generated by them admit a reasonable description. A motivation to study an algebra generated by Toeplitz operators (rather then just Toeplitz operators themselves) lies, first, in a possibility to apply more tools, in particular those coming from the algebraic toolbox. Secondly, the results obtained are applicable not only for generating Toeplitz operators but for all elements of the algebra.

A fundamental result due to Coburn [7], going back to the early 70th, describes the structure of the C^*-algebra generated by Toeplitz operators with $C(\overline{\mathbb{B}^n})$-symbols. This work initiated an extensive study of algebras generated by Toeplitz operators with symbols from certain predefined classes. The majority of the results obtained deal with Toeplitz operators that act on the Bergman space on the unit disk. The multidimensional setting, even the case of the unit ball, is more difficult as, beyond the class of continuous symbols, the symbol-functions may have more sophisticated behavior than for the one-dimensional case of the unit disk.

In this paper we study algebras generated by Toeplitz operators which act on weighted Bergman spaces over the complex two-dimensional unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$. Here the dimension $n = 2$ of the underlying domain is minimal such that the proposed approach is meaningful. Discussing this lowest dimensional case permits us to present the main ideas in a more simple and transparent form not burdened with technical details. However, a similar approach can be applied in the higher dimensional framework in which some new features are present. Detailed study of the higher dimensional case will be presented in forthcoming publications.

It has been observed in [17] that Toeplitz operators, with symbols invariant under the action of the (maximal Abelian) subgroup $\mathbb{T}^2$ of all biholomorphisms of $\mathbb{B}^2$, generate a commutative C^*-algebra on any weighted Bergman space $\boldsymbol{\mathcal{A}}^2_\lambda(\mathbb{B}^2)$. In this case there exists a unitary operator R_λ that maps $\boldsymbol{\mathcal{A}}^2_\lambda(\mathbb{B}^2)$ onto the one-sided sequence space $\ell_2 = \ell_2(\mathbb{Z}_+)$ (the direct sum of one-dimensional Hilbert spaces $\mathbb{C}$). For Toeplitz operators $\boldsymbol{T}^\lambda_a$ with bounded and group invariant symbols $a = a(|z_1|, |z_2|)$, these one-dimensional spaces $\mathbb{C}$ are invariant for the operator $R_\lambda \boldsymbol{T}^\lambda_a R^*_\lambda$. In particular, $R_\lambda \boldsymbol{T}^\lambda_a R^*_\lambda$ acts on each of these spaces as multiplication by a constant operator, and the commutativity result trivially follows.

In the present paper we consider symbols that are invariant under the action of the subgroup $\{1\} \times \mathbb{T} \cong \mathbb{T}$ of $\mathbb{T}^2$. On the one hand replacing $\mathbb{T}^2$ by a strict subgroup enlarges