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Laplace transforms and valuations

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It is proved that the classical Laplace transform is a continuous valuation which is positively GL(n) covariant and logarithmic translation covariant. Conversely, these properties turn out to be sufficient to characterize this transform.

1. Introduction

Let $f : [0, \infty) \to \mathbb{R}$ be a measurable function. The *Laplace transform* of f is given by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)dt, \quad s \in \mathbb{R}$$

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whenever the integral converges. In the 18th century, Euler first considered this transform to solve second-order linear ordinary differential equations with constant coefficients. One hundred years later, Petzval and Spitzer named this transform after Laplace. Doetsch initiated systematic investigations in 1920s. The Laplace transform now is widely used for solving ordinary and partial differential equations. Therefore, it is a useful tool not only for mathematicians but also for physicists and engineers (see, for example, [7]).

The Laplace transform has been generalized to the multidimensional setting in order to solve ordinary and partial differential equations in boundary value problems of several variables (see, for example, [6]). Let f be a compactly supported function that belongs to $L^1(\mathbb{R}^n)$. The multidimensional Laplace transform of f is defined as

$$\mathcal{L}f(x) = \int\limits_{\mathbb{R}^n} f(y)e^{-x\cdot y}dy, \quad x \in \mathbb{R}^n.$$

The Laplace transform is also considered on $\mathcal{K}^n_n$, the set of n dimensional convex bodies (i.e., compact convex sets) in $\mathbb{R}^n$. The *Laplace transform* of $K \in \mathcal{K}^n_n$ is defined by

$$\mathcal{L}K(x) = \mathcal{L}(\mathbb{1}_K)(x) = \int\limits_K e^{-x\cdot y}dy, \quad x \in \mathbb{R}^n,$$

where $\mathbb{1}_K$ is the indicator function of K. Making use of the logarithmic version of this transform, Klartag [19] improved Bourgain's estimate on the slicing problem (or hyperplane conjecture), which is one of the main open problems in the asymptotic theory of convex bodies. It asks whether every convex body of volume 1 has a hyperplane section through the origin whose volume is greater than a universal constant (see also [20] for more information).

Noticing that both Laplace transforms are valuations, we aim at a deeper understanding on these classical integral transforms. A function z defined on a lattice $(\Gamma, \vee, \wedge)$ and taking values in an abelian semigroup is called a *valuation* if

$$z(f \vee g) + z(f \wedge g) = z(f) + z(g) \tag{1.1}$$

for all $f, g \in \Gamma$. A function z defined on some subset Γ_0 of Γ is called a valuation on Γ_0 if (1.1) holds whenever $f, g, f \vee g, f \wedge g \in \Gamma_0$. Valuations were a key ingredient in Dehn's solution of Hilbert's Third Problem in 1901. They are closely related to dissections and lie at the very heart of geometry. Here, valuations were considered on the space of convex bodies in $\mathbb{R}^n$, denoted by $\mathcal{K}^n$. Perhaps the most famous result is Hadwiger's characterization theorem which classifies all continuous and rigid motion invariant real valued valuations on $\mathcal{K}^n$. Klain [15] provided a shorter proof of this beautiful result based on the following characterization of the volume.