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Energy identity for approximate harmonic maps from surfaces to general targets

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A B S T R A C T

Let u_n be a sequence of mappings from a closed Riemannian surface M to a general Riemannian manifold N. If u_n satisfies

$$\sup_n \big(\|\nabla u_n\|_{L^2(M)} + \|\tau(u_n)\|_{L^p(M)} \big) \le \Lambda \quad \text{for some } p > 1,$$

where $\tau(u_n)$ is the tension field of u_n, then the so called energy identity and neckless property hold during blowing up. This result is sharp by Parker's example, where the tension fields of the mappings from Riemannian surface are bounded in $L^1(M)$ but the energy identity fails.

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1. Introduction

Let (M,g) be a closed Riemannian manifold and (N,h) be a compact Riemannian manifold without boundary. Let u be a mapping from M to N in $W^{1,2}(M,N)$. We define the Dirichlet energy of u as follows

$$E(u) = \int_M e(u)dV,$$

where dV is the volume element of (M,g), and $e(u)$ is the density of u

$$e(u) = \frac{1}{2}|du|^2 = \text{Trace}_g u^* h,$$

where u^*h is the pull-back of the metric tensor h.

A map $u \in C^1(M,N)$ is called harmonic if it is a critical point of the energy E. By the Nash embedding theorem, (N,h) can be isometrically embedded into a Euclidean space $\mathbb{R}^k$ for some positive integer k with the metric induced from the Euclidean metric. Hence, a map $u \in C^1(M,N)$ can be viewed as a map of $C^1(M,\mathbb{R}^k)$ whose image lies in N. Then we can obtain the Euler–Lagrange equation

$$\triangle u - A(u)(du,du) = 0, \quad \text{or} \quad P(u)\triangle u = 0, \tag{1.1}$$

where $A(u)(du,du)$ is the second fundamental form of N in $\mathbb{R}^k$. Let $P(y) : \mathbb{R}^k \to T_yN$ be the orthogonal projection map. The tension field $\tau(u)$ is defined by

$$\tau(u) \stackrel{\text{def}}{=} \triangle u - A(u)(du,du) = P(u)\triangle u. \tag{1.2}$$

Then u is harmonic if and only if $\tau(u) = 0$. We refer to [8] for the systematic study of the harmonic maps.

Harmonic maps are of special interest when M is a Riemannian surface, because the Dirichlet energy is conformally invariant in two dimensions. It is an important question to understand the limiting behavior of sequences of harmonic maps. Let u_n be a sequence of mappings from a Riemannian surface M to N with bounded energy. It is clear that u_n (up to a subsequence) converges weakly to u in $W^{1,2}(M,N)$ for some $u \in W^{1,2}(M,N)$. In general, it may not converge strongly in $W^{1,2}(M,N)$ due to the concentration of the energy at finitely many points [14]. Thus, it is natural to ask (1) whether the lost energy is exactly the sum of energies of some harmonic spheres (bubbles), which are defined as harmonic maps from $\mathbb{S}^2$ to N; (2) whether attaching all possible bubbles to the weak limit gives uniform convergence. The first property is the so called energy identity, and the second one is called the bubble tree convergence.