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Maximum of the resolvent over matrices with given spectrum

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In numerical analysis it is often necessary to estimate the condition number $CN(T) = \|T\| \cdot \|T^{-1}\|$ and the norm of the resolvent $\|(\zeta - T)^{-1}\|$ of a given $n \times n$ matrix T. We derive new spectral estimates for these quantities and compute explicit matrices that achieve our bounds. We recover the fact that the supremum of $CN(T)$ over all matrices with $\|T\| \leq 1$ and minimal absolute eigenvalue $r = \min_{\lambda \in \sigma(T)} |\lambda| > 0$ is the Kronecker bound $\frac{1}{r^n}$. This result is subsequently generalized by computing for given ζ in the closed unit disc the supremum of $\|(\zeta - T)^{-1}\|$, where $\|T\| \leq 1$ and the spectrum $\sigma(T)$ of T is constrained to remain at a pseudo-hyperbolic distance of at least $r \in (0, 1]$ around ζ. We find that the supremum is attained by a triangular Toeplitz matrix. This provides a simple class of structured matrices on which condition numbers and resolvent norm bounds can be studied numerically. The occurring Toeplitz matrices are so-called model matrices, i.e. matrix representations of the compressed

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backward shift operator on the Hardy space H_2 to a finite-dimensional invariant subspace.

1. Introduction

Let $\mathcal{M}_n$ be the set of complex $n \times n$ matrices and let $\|T\|$ denote the operator norm of $T \in \mathcal{M}_n$. We denote by $\sigma = \sigma(T)$ the spectrum of T and by m_T its minimal polynomial. We denote by $|m_T|$ the degree of m_T. In this article we are interested in estimates of the type

$$\|R(\zeta, T)\| \leq \Phi(|m_T|, \sigma, \zeta) \tag{1}$$

where $R(\zeta, T) = (\zeta - T)^{-1}$ denotes the resolvent of T at point $\zeta \in \mathbb{C} - \sigma$ and Φ is a function of $|m_T|$, σ and ζ. We will think of an estimate of the above type as an extremal problem. That is under certain constraints on the set of admissible matrices T and fixed ζ, we are looking to maximize $\|R(\zeta, T)\|$ over this set. In our discussion we will generally impose that $\|T\| \leq 1$ and call such T a *contraction*. Note that this condition can always be achieved by normalization $T/\|T\|$. $\mathcal{C}_n \subset \mathcal{M}_n$ denotes the set of contractions. To ensure that the resolvent is finite, ζ must be separated from σ. Depending on the situation it will be convenient to measure this separation in *Euclidean* distance

$$d(z, w) := |z - w| \ , \ z, w \in \mathbb{C}$$

or *pseudo-hyperbolic* distance

$$p(z, w) := \left| \frac{z - w}{1 - \overline{z}w} \right|, \ \ z, w \in \mathbb{D},$$

where $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ is the open unit disc in the complex plane. We will also write $d(z, \sigma)$ and $p(z, \sigma)$ for the respective distance of z from a set $\sigma \subset \mathbb{C}$. We present a method that allows to compute the best possible function Φ when σ is constrained to remain at a pseudo-hyperbolic distance of at least $r \in (0, 1]$ around ζ. Our approach naturally yields a class of "worst" matrices for this problem. As it turns out for any $\zeta \notin \sigma$ among these matrices are so-called analytic Toeplitz matrices. Generally Toeplitz matrices are characterized by the existence of a sequence of complex numbers $a = (a_k)_{k=-n+1}^{k=n-1}$ with

$$T = \begin{pmatrix} a_0 & a_{-1} & . & . & a_{-n+1} \\ a_1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & a_{-1} \\ a_{n-1} & . & . & a_1 & a_0 \end{pmatrix}.$$