Journal of Functional Analysis 273 (2017) 3327–3405

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Kantorovich duality for general transport costs and applications ☆

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A R T I C L E   I N F O

Article history:
Received 13 January 2016
Accepted 23 August 2017
Available online 4 September 2017
Communicated by M. Ledoux

MSC:
60E15
32F32
26D10

Keywords:
Duality
Transport inequalities
Logarithmic-Sobolev inequalities
Metric spaces

A B S T R A C T

We introduce a general notion of transport cost that encompasses many costs used in the literature (including the classical one and weak transport costs introduced by Talagrand and Marton in the 90's), and prove a Kantorovich type duality theorem. As a by-product we obtain various applications in different directions: we give a short proof of a result by Strassen on the existence of a martingale with given marginals, we characterize the associated transport-entropy inequalities together with the log-Sobolev inequality restricted to convex/concave functions. We also provide explicit examples of discrete measures satisfying the weak transport-entropy inequalities derived here.

☆ Supported by the grants ANR 2011 BS01 007 01, ANR 10 LABX-58, ANR11-LBX-0023-01; the last author is supported by the NSF grants DMS 1101447 and 1407657, and is also grateful for the hospitality of Université Paris Est Marne La Vallée. All authors acknowledge the kind support of the American Institute of Mathematics (AIM, California).

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http://dx.doi.org/10.1016/j.jfa.2017.08.015
0022-1236/

Contents

1. Introduction

Concentration of measure phenomenon was introduced in the seventies by V. Milman [46] in his study of asymptotic geometry of Banach spaces. It was then studied in depth by many authors including Gromov [32,31], Talagrand [63], Maurey [44], Ledoux [36,10], Bobkov [6,11] and many others and played a decisive role in analysis, probability and statistics in high dimensions. We refer to the monographs [37] and [16] for an overview of the field.

One classical example of such a phenomenon can be observed for the standard Gaussian measure γ_m on $\mathbb{R}^m$. It follows from the well-known Sudakov–Tsirelson–Borell isoperimetric result in Gauss space [62,15] that if $X_1, \dots, X_n$ are n i.i.d random vectors with law γ_m and $f : (\mathbb{R}^m)^n \to \mathbb{R}$ is a 1-Lipschitz function (with respect to the Euclidean norm), then

$$\mathbb{P}(f(X_1, \dots, X_n) > \mathrm{med}(f) + t) \leq e^{-(t-t_o)^2/(2a)}, \qquad \forall t \geq t_o, \tag{1.1}$$