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# Quasilinear elliptic equations on noncompact Riemannian manifolds ☆

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ABSTRACT

The existence of solutions to a class of quasilinear elliptic problems on noncompact Riemannian manifolds, with finite volume, is investigated. Boundary value problems, with homogeneous Neumann conditions, in possibly irregular Euclidean domains are included as a special instance. A nontrivial solution is shown to exist under an unconventional growth condition on the right-hand side, which depends on the geometry of the underlying manifold. The identification of the critical growth is a crucial step in our analysis, and entails the use of the isocapacitary function of the manifold. A condition involving its isoperimetric function is also provided.



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## 1. Introduction

The present paper is concerned with the existence of solutions to semilinear elliptic equations on an $n$-dimensional Riemannian manifold $M$, whose weak formulation reads

$$\int_M |\nabla u|^{p-2}\nabla u\cdot\nabla v\,d\mathcal{H}^n=\int_M f(u)\,v\,d\mathcal{H}^n \tag{1.1}$$

for every test function $v$ in the Sobolev space $W^{1,p}(M)$. Here, $p\in(1,\infty)$, $\nabla$ stands for the gradient operator on $M$, $|\nabla u|$ denotes its length, determined by the scalar product "$\cdot$" associated with the Riemannian metric on $M$, and $\mathcal{H}^n$ is the volume measure on $M$ induced by the metric. The function $f:\mathbb{R}\to\mathbb{R}$ is continuous, and satisfies suitable growth conditions for the right-hand side of (1.1) to be well defined for every test function $v$.

Throughout, we assume that $M$ is connected, without boundary, and

$$\mathcal{H}^n(M)<\infty \tag{1.2}$$

Although compact manifolds are included as a special case, the main emphasis will be on the noncompact case. Its treatment calls for new inequalities of Sobolev type, whose form is patterned on the geometry of $M$.

Equation (1.1) encompasses problems of diverse nature, depending on analytic-geometric properties of $M$. Their common feature is that of being the Euler equation of the functional defined as

$$J(u)=\frac{1}{p}\int_M |\nabla u|^p\,d\mathcal{H}^n-\int_M F(u)d\mathcal{H}^n \tag{1.3}$$

for $u\in W^{1,p}(M)$, where $F:\mathbb{R}\to\mathbb{R}$ is the primitive of $f$ given by

$$F(t)=\int_0^t f(r)\,dr\quad\text{for } t\in\mathbb{R}. \tag{1.4}$$

Hence, the solutions to (1.1) are the critical points of the functional $J$. For instance, if the space of smooth compactly supported functions on $M$ is dense in $W^{1,p}(M)$ – this certainly holds when $M$ is a complete Riemannian manifold – then (1.1) corresponds to the weak form of the equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u)=f(u)\qquad\text{on } M. \tag{1.5}$$

In the case when $M$ is an open subset $\Omega$ of a Riemannian manifold, and in particular of the Euclidean space $\mathbb{R}^n$, equation (1.1) amounts to the definition of weak solution to the Neumann boundary value problem