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A nonlinear free boundary problem with a self-driven Bernoulli condition ☆

Serena Dipierro [a,*], Aram Karakhanyan [b], Enrico Valdinoci [c,a,d]

[a] *Dipartimento di Matematica Federigo Enriques, Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy*
[b] *Maxwell Institute for Mathematical Sciences and School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom*
[c] *School of Mathematics and Statistics, University of Melbourne, Richard Berry Building, Parkville, VIC 3010, Australia*
[d] *Istituto di Matematica Applicata e Tecnologie Informatiche, Consiglio Nazionale delle Ricerche, Via Ferrata 1, 27100 Pavia, Italy*

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A B S T R A C T

We study a Bernoulli type free boundary problem with two phases

$$J[u] = \int_\Omega |\nabla u(x)|^2\, dx + \Phi\big(\mathcal{M}_-(u), \mathcal{M}_+(u)\big),$$
$$u - \bar{u} \in W_0^{1,2}(\Omega),$$

where $\bar{u} \in W^{1,2}(\Omega)$ is a given boundary datum. Here, $\mathcal{M}_1$ and $\mathcal{M}_2$ are weighted volumes of $\{u \leqslant 0\} \cap \Omega$ and $\{u > 0\} \cap \Omega$, respectively, and Φ is a nonnegative function of two real variables.

We show that, for this problem, the Bernoulli constant, which determines the gradient jump condition across the free boundary, is of global type and it is indeed determined by the weighted volumes of the phases.

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* Corresponding author.
E-mail addresses: serena.dipierro@unimi.it (S. Dipierro), aram.karakhanyan@ed.ac.uk (A. Karakhanyan), enrico@math.utexas.edu (E. Valdinoci).

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In particular, the Bernoulli condition that we obtain can be seen as a pressure prescription in terms of the volume of the two phases of the minimizer itself (and therefore it depends on the minimizer itself and not only on the structural constants of the problem).

Another property of this type of problems is that the minimizer in Ω is not necessarily a minimizer in a smaller subdomain, due to the nonlinear structure of the problem.

Due to these features, this problem is highly unstable as opposed to the classical case studied by Alt, Caffarelli and Friedman. It also interpolates the classical case, in the sense that the blow-up limits of u are minimizers of the Alt–Caffarelli–Friedman functional. Namely, the energy of the problem somehow linearizes in the blow-up limit.

As a special case, we can deal with the energy levels generated by the volume term $\Phi(0, r_2) \simeq r_2^{\frac{n-1}{n}}$, which interpolates the Athanasopoulos–Caffarelli–Kenig–Salsa energy, thanks to the isoperimetric inequality.

In particular, we develop a detailed optimal regularity theory for the minimizers and for their free boundaries.

Contents

1. Introduction

After [2–4], a classical problem in the free boundary theory consists in studying the minimizers of an energy functional which is the *linear superposition* of a Dirichlet energy and a volume term. In this case, minimizers are proved to be harmonic away from the free boundary. Also, minimizers naturally enjoy a free boundary condition which can be seen as a balance of the normal derivatives across the interface.

This type of problems has a natural interpretation in terms of two dimensional flows of two irrotational, incompressible and inviscid fluids. Indeed, if the fluids have velocities