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On vanishing near corners of transmission eigenfunctions

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ABSTRACT

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $V \in L^\infty(\Omega)$ be a potential function. Consider the following transmission eigenvalue problem for nontrivial $v, w \in L^2(\Omega)$ and $k \in \mathbb{R}_+$,

$$\begin{cases} (\Delta + k^2)v = 0 & \text{in } \Omega, \\ (\Delta + k^2(1+V))w = 0 & \text{in } \Omega, \\ w - v \in H_0^2(\Omega), \quad \|v\|_{L^2(\Omega)} = 1. \end{cases}$$

We show that the transmission eigenfunctions v and w carry the geometric information of $\mathrm{supp}(V)$. Indeed, it is proved that v and w vanish near a corner point on $\partial\Omega$ in a generic situation where the corner possesses an interior angle less than π and the potential function V does not vanish at the corner point. This is the first quantitative result concerning the intrinsic property of transmission eigenfunctions and enriches the classical spectral theory for Dirichlet/Neumann Laplacian. We also discuss its implications to inverse scattering theory and invisibility.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $V \in L^\infty(\Omega)$ be a potential function. Consider the following (interior) transmission eigenvalue problem for $v, w \in L^2(\Omega)$,

$$\begin{cases} (\Delta + k^2)v = 0 & \text{in } \Omega, \\ (\Delta + k^2(1+V))w = 0 & \text{in } \Omega, \\ w - v \in H_0^2(\Omega), \quad \|v\|_{L^2(\Omega)} = 1. \end{cases} \tag{1.1}$$

If the system (1.1) admits a pair of nontrivial solutions (v, w), then k is referred to as an *(interior) transmission eigenvalue* and (v, w) is the corresponding pair of *(interior) transmission eigenfunctions*. Note in particular that nothing is imposed a-priori on the boundary values of v or w individually. In this paper, we are mainly interested in the real eigenvalues, $k \in \mathbb{R}_+$, which are physically relevant. The study of the transmission eigenvalue problem has a long history and is of significant importance in scattering theory. The transmission eigenvalue problem is a type of non elliptic and non self-adjoint problem, so its study is mathematically interesting and challenging. In the literature, the existing results are mainly concerned with the spectral properties of the transmission eigenvalues, including the existence, discreteness and infiniteness, and Weyl laws; see for example [4,7,11,25,30–32] and the recent survey [8]. There are few results concerning the intrinsic properties of the transmission eigenfunctions. Here we are aware that the completeness of the set of *generalized transmission eigenfunctions* in L^2 is proven in [4,31].

In this paper, we are concerned with the vanishing properties of *interior transmission eigenfunctions*. It is shown that in admissible geometric situations, transmission eigenfunctions which can be approximated suitably by Herglotz waves will vanish at corners of the support of the potential V. To our best knowledge, this is the first quantitative result on intrinsic properties of transmission eigenfunctions. As expected, these carry geometric information of the support of the underlying potential V as well as other interesting consequences and implications in scattering theory, which we shall discuss in more details in Section 7.

The location of vanishing of eigenfunctions is an important area of study in the classical spectral theory for the Dirichlet/Neumann Laplacian. Two important topics are the *nodal sets* and *eigenfunction localization*. The former is the set of points in the domain where the eigenfunction vanishes. For the latter, an eigenfunction is said to be *localized* if most of its L^2-energy is contained in a subdomain which is a fraction of the total domain. Considerable effort has been spent on the nodal sets and localization in the classical spectral theory. We refer to the recent survey [17]. For the curious, we mention briefly basic facts about them, all of which are completely open for transmission eigenfunctions. Nodal sets are C^∞-curves whose intersections form equal angles. By the celebrated Courant's nodal line theorem, the nodal set of the m-th eigenfunction divides the domain into at most m nodal domains. Localization seems to be a more recent topic