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CONDITIONALLY BI-FREE INDEPENDENCE FOR PAIRS OF FACES

YINZHENG GU AND PAUL SKOUFRANIS

Abstract. In this paper, the notion of conditionally bi-free independence for pairs of faces is introduced. The notion of conditional (ℓ, r)-cumulants is introduced and it is demonstrated that conditionally bi-free independence is equivalent to the vanishing of mixed cumulants. Furthermore, limit theorems for the additive conditionally bi-free convolution are studied using both combinatorial and analytic techniques. In particular, a conditionally bi-free partial $\mathcal{R}$-transform is constructed and a conditionally bi-free analogue of the Lévy-Hinčin formula for planar Borel probability measures is derived.

1. Introduction

The basic framework in non-commutative probability theory is a pair $(\mathcal{A}, \psi)$, called a non-commutative probability space, where $\mathcal{A}$ is a (complex) unital algebra and ψ is a unital linear functional on $\mathcal{A}$. Subalgebras of $\mathcal{A}$ are said have a certain independence with respect to ψ if there is a specific rule of calculating the joint distributions. There are several important notions of independence in the literature. According to [14, 15] there are exactly five notions of universal/natural independence: classical, free, Boolean, monotone, and anti-monotone. These notions of independence have very similar theories such as the combinatorics of cumulants and the analytic aspects of convolutions on probability measures. On the other hand, the notion of conditionally free independence was introduced and studied in [4, 5] as a notion of independence with respect to a pair of unital linear functionals (φ, ψ) on a unital algebra $\mathcal{A}$. Although mainly intended as a generalization of free independence, it turned out (see [4, 8]) that Boolean and monotone independences, especially their relative convolutions, can also be unified in terms of conditionally free independence.

Free probability for pairs of faces, or bi-free probability for short, is a generalization of free probability introduced by Voiculescu [21] in order to study the non-commutative left and right actions of algebras on a reduced free product space simultaneously. Again, the basic framework is a non-commutative probability space $(\mathcal{A}, \psi)$, but the corresponding independence, called bi-free independence, is defined for pairs of subalgebras of $\mathcal{A}$ instead. Since its inception, bi-free probability has received a lot of attention as many old results from free probability have been extended to the bi-free setting and new results have been developed. In particular, it was noticed in [21] that both classical and free independences can be viewed as specific cases of bi-free independence and it was noticed in [17] that Boolean and monotone independences also occur in bi-free probability. Thus bi-free probability is in a certain sense another unifying theory. It is then natural to combine the two mentioned unifying theories together and develop a notion of conditionally bi-free independence, which is the main focus of this paper.

This paper contains six sections, including this introduction, which are structured as follows. In Section 2, basic notions and results from bi-free and conditionally free probability theories are recalled, with an emphasis on the combinatorial aspects.

In Section 3, two notions of conditionally bi-free independence are provided. The first arises naturally by combining the constructions of bi-free and conditionally free independences. The second is defined as the vanishing of certain cumulants. More precisely, as bi-free independence can be characterized by the vanishing of mixed (ℓ, r)-cumulants, and as conditionally free independence can be characterized by the vanishing of mixed free and c-free cumulants, we introduce the family of c-(ℓ, r)-cumulants and define combinatorially c-bi-free independence as the vanishing of mixed (ℓ, r)- and c-(ℓ, r)-cumulants.

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