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Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance

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A B S T R A C T

We study a class of ergodic quantum Markov semigroups on finite-dimensional unital C^*-algebras. These semigroups have a unique stationary state σ, and we are concerned with those that satisfy a quantum detailed balance condition with respect to σ. We show that the evolution on the set of states that is given by such a quantum Markov semigroup is gradient flow for the relative entropy with respect to σ in a particular Riemannian metric on the set of states. This metric is a non-commutative analog of the 2-Wasserstein metric, and in several interesting cases we are able to show, in analogy with work of Otto on gradient flows with respect to the classical 2-Wasserstein metric, that the relative entropy is strictly and uniformly convex with respect to the Riemannian metric introduced here. As a consequence, we obtain a number of new inequalities for the decay of relative entropy for ergodic quantum Markov semigroups with detailed balance.

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1. Introduction

Let $\mathcal{A}$ be a finite-dimensional C^*-algebra with unit $\mathbf{1}$. We may identify $\mathcal{A}$ with a C^*-subalgebra of $\mathcal{M}_n(\mathbb{C})$, the C^*-algebra of $n \times n$ matrices, for some n. In finite dimension, there is no difference between weak and norm closure, and so $\mathcal{A}$ is also a von Neumann algebra. A *Quantum Markov Semigroup* (QMS) is a continuous one-parameter semigroup of linear transformations $(\mathscr{P}_t)_{t\geq 0}$ on $\mathcal{A}$ such that for each $t \geq 0$, $\mathscr{P}_t$ is completely positive and $\mathscr{P}_t\mathbf{1} = \mathbf{1}$. Associated to any QMS $\mathscr{P}_t = e^{t\mathscr{L}}$, is the dual semigroup $\mathscr{P}_t^\dagger$ acting on $\mathfrak{S}_+(\mathcal{A})$, the set of faithful states of $\mathcal{A}$. (When there is no ambiguity, we simply write $\mathfrak{S}_+$.) The QMS $\mathscr{P}_t$ is *ergodic* in case $\mathbf{1}$ spans the eigenspace of $\mathscr{P}_t$ for the eigenvalue 1. In that case, there is a unique invariant state σ. While σ need not be faithful, a natural projection operation allows us to assume, effectively without loss of generality, that $\sigma \in \mathfrak{S}_+(\mathcal{A})$. Characterizations of the generators of quantum Markov semigroups on the C^*-algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices were given at the same time by Gorini, Kossakowski and Sudershan [30], and by Lindblad [44] in a more general setting (but still assuming norm continuity of the semigroup). Such semigroups are often called Lindblad semigroups.

The notion of *detailed balance* in the theory of classical Markov processes has several different quantum counterparts, as discussed below. One of these is singled out here, with a full discussion of how it relates to other variants and why it is physically natural. Suffice it to say here that, as we shall see, the class of ergodic QMS that satisfy the detailed balance condition includes a wide variety of examples arising in physics.

The set of faithful states $\mathfrak{S}_+(\mathcal{A})$ may be identified with the set of invertible density matrices σ on $\mathbb{C}^n$ that belong to $\mathcal{A}$, as is recalled below. For $\rho, \sigma \in \mathfrak{S}_+(\mathcal{A})$, the *relative entropy of ρ with respect to σ* is the functional

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]\ . \tag{1.1}$$

We show in Theorem 7.6 that associated to any QMS $\mathscr{P}_t = e^{t\mathscr{L}}$ satisfying detailed balance, there is a Riemannian metric $g_{\mathscr{L}}$ on $\mathfrak{S}_+$ such that the flow on $\mathfrak{S}_+$ induced by the dual semigroup $\mathscr{P}_t^\dagger$ is *gradient flow* for the metric $g_{\mathscr{L}}$ of the relative entropy $D(\cdot\|\sigma)$ with respect to the invariant state $\sigma \in \mathfrak{S}_+$. In several cases, we shall show that the relative entropy functional is geodesically convex for $g_{\mathcal{L}}$, and as a consequence, we shall deduce a number of functional inequalities that are useful for studying the evolution governed by $\mathscr{P}_t$. In particular, we shall deduce several sharp relative entropy dissipation inequalities. Some of these are new; see e.g. Theorem 8.5 and Theorem 8.6.

The Riemannian distance corresponding to $g_{\mathscr{L}}$ will be seen to be a very natural analog of the 2-Wasserstein distance on the space of probability densities on $\mathbb{R}^n$ [70, Chapter 6]. Otto showed [53] that a large number of classical evolution equations could be viewed as gradient flow in the 2-Wasserstein metric for certain functionals, and that when the functionals were geodesically uniformly convex for this geometry, a host of useful functional inequalities were consequently valid. This is for instance the case when the functional

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