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Quantum measurable cardinals

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ABSTRACT

We investigate states on von Neumann algebras which are not normal but enjoy various forms of infinite additivity, and show that these exist on $B(H)$ if and only if the cardinality of an orthonormal basis of H satisfies various large cardinal conditions. For instance, there is a singular countably additive pure state on $B(l^2(\kappa))$ if and only if κ is Ulam measurable, and there is a singular $<\kappa$-additive pure state on $B(l^2(\kappa))$ if and only if κ is measurable. The proofs make use of Farah and Weaver's theory of quantum filters [12]. We can generalize some of these characterizations to arbitrary von Neumann algebras. Applications to Ueda's peak set theorem for von Neumann algebras are discussed in the final section.

1. Measurable cardinals

In a recent paper [9] Blecher and Labuschagne investigated whether every von Neumann algebra verifies Ueda's peak set theorem [22]. It was discovered that the answer

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turns on the existence of singular states with a certain continuity property. Such states, which falsify the von Neumann algebra version of Ueda's theorem, exist on $l^\infty(\kappa)$ and $B(l^2(\kappa))$ provided κ has a large cardinal property related to measurability. (Throughout this paper κ denotes an infinite cardinal.) Similar conditions on states were studied in the context of axiomatic von Neumann algebra quantum mechanics in e.g. [13,10]. This motivated us to consider the general question of the existence of singular states on $B(H)$ with various continuity properties. We will return to Ueda's theorem in the final section of the paper.

In set theory there is an elaborate hierarchy of "large cardinal" properties, some of which involve various notions of measurability [11,15]. Four of these are of primary interest to us here. An uncountable cardinal κ is said to be

- *measurable* if there is a nonzero $<\kappa$-additive $\{0, 1\}$-valued measure on κ which vanishes on singletons
- *real-valued measurable* if there is a $<\kappa$-additive probability measure on κ which vanishes on singletons
- *Ulam measurable* if there is a nonzero countably additive $\{0, 1\}$-valued measure on κ which vanishes on singletons
- *Ulam real-valued measurable* if there is a countably additive probability measure on κ which vanishes on singletons.

Here a "measure on κ" is understood to be defined on all subsets of κ, and "$<\kappa$-additive" means "additive on any family of fewer than κ disjoint sets". In a set theory context the latter property would normally be called "κ-additive", but that becomes awkward in the case of countable additivity, so we shall use this slightly modified terminology.

No cardinal of any of these types can be proven to exist in ordinary set theory, assuming ordinary set theory is consistent. They are "large" in the sense that the smallest real-valued measurable or Ulam real-valued measurable cardinal, if one exists, must be weakly inaccessible, and the smallest measurable or Ulam measurable cardinal, if one exists, must be strongly inaccessible. It is generally believed that the existence of such cardinals is consistent with ordinary set theory. However, this, if true, could not be proven within ordinary set theory. What we can say is that they all have the same consistency strength, i.e., the consistency of any of the theories

ZFC + "a measurable cardinal exists"
ZFC + "a real-valued measurable cardinal exists"
ZFC + "an Ulam measurable cardinal exists"
ZFC + "an Ulam real-valued measurable cardinal exists"

implies the consistency of each of the others.

We summarize the basic relations among these notions. Every measurable cardinal is real-valued measurable, and among cardinals $> 2^{\aleph_0}$ the two notions coincide. However,

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