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A Hele–Shaw problem for tumor growth

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A B S T R A C T

We consider weak solutions to a problem modeling tumor growth. Under certain conditions on the initial data, solutions can be obtained by passing to the stiff (incompressible) limit in a porous medium type problem with a Lotka–Volterra source term describing the evolution of the number density of cancerous cells. We prove that such limit solutions solve a free boundary problem of Hele–Shaw type. We also obtain regularity properties, both for the solution and for its free boundary.

The main new difficulty arises from the competition between the growth due to the source, which keeps the initial singularities, and the free boundary which invades the domain with a regularizing effect. New islands can be generated at singular times.

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1. Introduction

We consider solutions (n, p) to the nonlinear parabolic equation

$$\partial_t n = \Delta p + nG(p) \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times \mathbb{R}_+), \tag{1.1}$$

where the pair (n, p) lies in the so called Hele–Shaw graph,

$$0 \le n \le 1, \quad p \ge 0, \quad \text{and } p = 0 \quad \text{for } \; 0 \le n < 1, \tag{1.2}$$

with an initial data

$$n(\cdot, 0) = n^0 \in L^1_+(\mathbb{R}^N). \tag{1.3}$$

This problem was studied in [18] as a simple mechanical model for the propagation of tumors, following more elaborate models in [4,16]. In this setting n stands for the density of cancerous cells, and p for the pressure. The function G in the source term, taking account of pressure limited cell multiplication by division, satisfies

$$G \in C^1([0, \infty)), \quad G'(\cdot) < 0, \quad \text{there exists } p_M > 0 \text{ such that } G(p_M) = 0. \tag{1.4}$$

The threshold pressure p_M, sometimes called homeostatic pressure, is the smallest pressure that prevents cell multiplication because of contact inhibition. For simplicity, and without loss of generality, the maximum packing density of cells has been set to $n = 1$; see (1.2).

Some notations. We denote $Q = \mathbb{R}^N \times \mathbb{R}_+$, and, for $T > 0$, $Q_T = \mathbb{R}^N \times (0, T)$. Given $g : Q \to \mathbb{R}$, we will use several times the abridged notation $g(t)$ to describe the function $x \to g(x, t)$.

Whenever the initial data satisfy

$$n^0 \in BV(\mathbb{R}^N), \qquad 0 \le n^0 \le 1 \quad \text{a.e. in } \mathbb{R}^N, \tag{1.5}$$

existence of a solution to (1.1)–(1.3) can be proved by passing to the *stiff* (incompressible) limit $\gamma \to \infty$ for weak solutions (n_γ, p_γ) to the porous medium type problem

$$\partial_t n_\gamma - \operatorname{div}(n_\gamma \nabla p_\gamma) = n_\gamma G(p_\gamma) \quad \text{in } Q, \qquad n_\gamma(0) = n^0_\gamma \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where the density and the pressure are related by the law of state

$$p_\gamma = P_\gamma(n_\gamma), \qquad P_\gamma(n) = n^\gamma, \quad \gamma > 1, \tag{1.7}$$