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Endpoint Sobolev and BV continuity for maximal operators

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A B S T R A C T

In this paper we investigate some questions related to the continuity of maximal operators in $W^{1,1}$ and BV spaces, complementing some well-known boundedness results. Letting $\widetilde{M}$ be the one-dimensional uncentered Hardy–Littlewood maximal operator, we prove that the map $f \mapsto \big(\widetilde{M}f\big)'$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. In the discrete setting, we prove that $\widetilde{M} : BV(\mathbb{Z}) \to BV(\mathbb{Z})$ is also continuous. For the one-dimensional fractional Hardy–Littlewood maximal operator, we prove by means of counterexamples that the corresponding continuity statements do not hold, both in the continuous and discrete settings, and for the centered and uncentered versions.

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1. Introduction

The purpose of this paper is to present new results and discuss some open problems related to the continuity of the one-dimensional Hardy–Littlewood maximal operator in $W^{1,1}$ and BV spaces. Such continuity questions for maximal operators in Sobolev spaces are already nontrivial in the case $W^{1,p}$ with $p > 1$ (see [20,21]) and, to the best of our knowledge, this is the first time that the more subtle limiting case $p = 1$ is addressed in the literature. We start with a brief recollection of some of the recent developments on the general regularity theory for maximal operators.

1.1. Background

For $f \in L^1_{loc}(\mathbb{R}^d)$ we define the centered Hardy–Littlewood maximal function Mf by

$$Mf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int\limits_{B_r(x)} |f(y)| \, \mathrm{d}y \,, \tag{1.1}$$

where $B_r(x)$ is the open ball of center x and radius r, and $m(B_r(x))$ denotes its d-dimensional Lebesgue measure. The uncentered maximal function $\widetilde{M}f$ at a point x is defined analogously, taking the supremum of averages over open balls that contain the point x, but that are not necessarily centered at x. One of the pillars of harmonic analysis is the Hardy–Littlewood–Wiener theorem, which states that $M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a bounded operator when $1 < p \leq \infty$. The classical consequence of an L^p-bound for a maximal operator is the pointwise convergence (a.e.) of a particular sequence of associated objects (in this case, the integral averages as the radius goes to zero). Since M is sublinear, it follows directly from the boundedness statement that $M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is also a continuous map when $1 < p \leq \infty$. Similar results hold for $\widetilde{M}$.

An active topic of current research is the investigation of the regularity properties of maximal operators. One of the driving questions in this theory is whether a given maximal operator improves, preserves or destroys the *a priori* regularity of an initial datum f. With respect to Sobolev regularity, Kinnunen in the seminal paper [13] established that $M : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ is a bounded operator for $1 < p \leq \infty$ (the same holds for $\widetilde{M}$). Since M is not necessarily sublinear at the derivative level, the continuity of $M : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$, for $p > 1$, is a highly nontrivial problem, originally attributed to T. Iwaniec [12, Question 3]. This question was settled, affirmatively, by Luiro in [20], and this is perhaps the paper that most closely inspires the present work. Kinnunen's original result was later extended to a local setting in [14], to a fractional setting in [15] and to a multilinear setting in [9]. Other works on the regularity of maximal operators in Sobolev spaces and other function spaces include [1,16,18,23,25,28].

The understanding of the action of M on the endpoint space $W^{1,1}(\mathbb{R}^d)$ is a much more delicate issue. The question of whether the map $f \mapsto \nabla Mf$ is bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ was raised by Hajłasz and Onninen in [12] and remains unsolved in its full