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# On the geometry of projective tensor products

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In this work, we study the volume ratio of the projective tensor products $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ with $1 \le p \le q \le r \le \infty$. We obtain asymptotic formulas that are sharp in almost all cases. As a consequence of our estimates, these spaces allow for a nearly Euclidean decomposition of Kašin type whenever $1 \le p \le q \le r \le 2$ or $1 \le p \le 2 \le r \le \infty$ and $q = 2$. Also, from the Bourgain–Milman bound on the volume ratio of Banach spaces in terms of their cotype 2 constant, we obtain information on the cotype of these 3-fold projective tensor

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products. Our results naturally generalize to $k$-fold products $\ell_{p_1}^n \otimes_\pi \cdots \otimes_\pi \ell_{p_k}^n$ with $k \in \mathbb{N}$ and $1 \le p_1 \le \cdots \le p_k \le \infty$.


## 1. Introduction

In the geometry of Banach spaces the volume ratio $\mathrm{vr}(X)$ of an $n$-dimensional normed space $X$ is defined as the $n$-th root of the volume of the unit ball in $X$ divided by the volume of its John ellipsoid. This notion plays an important role in the local theory of Banach spaces and has significant applications in approximation theory. It formally originates in the works [28] and [29], which were influenced by the famous paper of B. Kašin [16] on nearly Euclidean orthogonal decompositions. Kašin discovered that for arbitrary $n \in \mathbb{N}$, the space $\ell_1^{2n}$ contains two orthogonal subspaces which are nearly Euclidean, meaning that their Banach–Mazur distance to $\ell_2^n$ is bounded by an absolute constant. S. Szarek [28] noticed that the proof of this result depends solely on the fact that $\ell_1^n$ has a bounded volume ratio with respect to $\ell_2^n$. In fact, it is essentially contained in the work of Szarek that if $X$ is a $2n$-dimensional Banach space, then there exist two $n$-dimensional subspaces each having a Banach–Mazur distance to $\ell_2^n$ bounded by a constant times the volume ratio of $X$ squared. This observation by S. Szarek and N. Tomczak-Jaegermann was further investigated in [29], where the concept of volume ratio was formally introduced, its connection to the cotype 2 constant of Banach spaces was studied, and Kašin type decompositions were proved for some classes of Banach spaces, such as the projective tensor product spaces $\ell_p^n \otimes_\pi \ell_2^n$, $1 \le p \le 2$.

Given two vector spaces $X$ and $Y$, their algebraic tensor product $X \otimes Y$ is the subspace of the dual space of all bilinear maps on $X \times Y$ spanned by elementary tensors $x \otimes y$, $x \in X$, $y \in Y$ (a formal definition is provided below). The theory of tensor products was established by A. Grothendieck in 1953 in his Résumé [13] and has a huge impact on Banach space theory (see, e.g., the survey paper [23]). This impact and the success of the concept of tensor products is to a large extent due to the work [17] of J. Lindenstrauss and A. Pełczynski in the late sixties who reformulated Grothendieck's ideas in the context of operator ideals and made this theory accessible to a broader audience. Today, tensor products appear naturally in numerous applications, among others, in the entanglement of qubits in quantum computing, in quantum information theory in terms of (random) quantum channels (e.g., [2–4,30]) or in theoretical computer science to represent locally decodable codes [11]. For an interesting and recently discovered connection between the latter and the geometry of Banach spaces we refer the reader to [6].

The geometry of tensor products of Banach spaces is complicated, even if the spaces involved are of simple geometric structure. For example, the 2-fold projective tensor product of Hilbert spaces, $\ell_2 \otimes_\pi \ell_2$, is naturally identified with the Schatten trace class $S_1$, the space of all compact operators $T : \ell_2 \to \ell_2$ equipped with the norm

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