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Tiling sets and spectral sets over finite fields ☆

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ABSTRACT

We study tiling and spectral sets in vector spaces over prime fields. The classical Fuglede conjecture in locally compact abelian groups says that a set is spectral if and only if it tiles by translation. This conjecture was disproved by T. Tao in Euclidean spaces of dimensions 5 and higher, using constructions over prime fields (in vector spaces over finite fields of prime order) and lifting them to the Euclidean setting. Over prime fields, when the dimension of the vector space is less than or equal to 2 it has recently been proven that the Fuglede conjecture holds (see [6]). In this paper we study this question in higher dimensions over prime fields and provide some results and counterexamples. In particular we prove the existence of spectral sets which do not tile in $\mathbb{Z}_p^5$ for all odd primes p and $\mathbb{Z}_p^4$ for all odd primes p such that $p \equiv 3 \bmod 4$. Although counterexamples in low dimensional groups over cyclic rings $\mathbb{Z}_n$ were previously known they were usually for non-prime n or a small, sporadic set of primes p rather than general constructions. This paper is a result of a Research Experience for Undergraduates program ran at the University of Rochester during the summer of 2015 by A. Iosevich, J. Pakianathan and G. Petridis.

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Contents

1. Introduction

The purpose of this paper is to study the relationships between tiling properties of sets and the existence of orthogonal exponential bases for functions on these sets in the context of vector spaces over finite fields.

A *tiling set* in a finite abelian group A is a set T such that A can be written as a disjoint union of translates $\{a + T | a \in B\}$ of the set T. The set B is the *tiling partner* of T. Note this means every element of A can be written uniquely as sum $t + b$, $t \in T$, $b \in B$. More generally for fixed $k \geq 1$, a set T is said to k-tile with k-tiling partner B if every element of A can be written as a sum $t + b, t \in T, b \in B$ in exactly k ways.

A subset $E \subseteq A$ is a called a *spectral set* if there exists a set of characters

$$\{\chi_b | b \in B\}$$

of A which forms an orthogonal basis of $L^2(E)$, the vector space of complex valued functions on E with Hermitian inner product $< f, g > = \sum_{e \in E} f(e)\bar{g}(e)$.

More technical definitions of these concepts are given in sections 3 and 4 of the paper.

As both tiling properties and spectral properties of sets depend only on the underlying abelian group of these vector spaces, it is enough to understand these relationships over prime fields $\mathbb{Z}_p = \mathbb{F}_p$. This is because for any prime p, the finite field $\mathbb{F}_{p^s}$ is additively isomorphic to $\mathbb{Z}_p^s$ so that $\mathbb{F}_{p^s}^d \cong \mathbb{Z}_p^{ds}$ as abelian groups. Due to this, in the remainder of this paper, we exclusively consider these questions in vector spaces over prime fields, i.e., in $\mathbb{Z}_p^d$, where p is a prime.