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The reconstruction theorem in Besov spaces

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A R T I C L E I N F O

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A B S T R A C T

The theory of regularity structures [9] sets up an abstract framework of *modelled distributions* generalising the usual Hölder functions and allowing one to give a meaning to several ill-posed stochastic PDEs. A key result in that theory is the so-called reconstruction theorem: it defines a continuous linear operator that maps spaces of modelled distributions into the usual space of distributions. In the present paper, we extend the scope of this theorem to analogues to the whole class of Besov spaces $\mathcal{B}^{\gamma}_{p,q}$ with non-integer regularity indices. We then show that these spaces behave very much like their classical counterparts by obtaining the corresponding embedding theorems and Schauder-type estimates.

Contents

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1. Introduction

The theory of regularity structures [9] provides an analytic framework which turns out to be powerful in providing solution theories to classes of singular parabolic stochastic PDEs. An important aspect of the theory is that, instead of describing the solution as an element of one of the classical spaces of functions/distributions, one provides a local description thereof as *generalised* Taylor polynomials attached to every space–time point. In the special case of smooth functions, this simply corresponds to Whitney's [15] interpretation of a Hölder function as the corresponding collection of usual Taylor polynomials associated to it. In the setting of stochastic PDEs, it is helpful to enrich the collection of usual monomials with some appropriate functionals built from the driving noise. Then, the solution of the stochastic PDE can (under some assumptions, of course) locally be expanded on this enlarged basis of monomials. In the case of some ill-posed stochastic PDEs, this procedure, or some closely related procedure as in [8], is already required to give a rigorous interpretation of what one even *means* for a (random) function/distribution to be a solution to the equation. We provide a more detailed presentation of the theory at the end of this introduction.

The original framework of the theory [9] used direct analogues to Hölder spaces of functions, but it turns out that this can be generalised to the whole class of Besov spaces and this is the main purpose of the present work. One motivation for this generalisation arose in a recent work on the construction of the solution of multiplicative stochastic heat equations starting from a Dirac mass at time 0, see [10]. Therein, we adapted the theory of regularity structures to $\mathcal{B}^{\alpha}_{p,\infty}$-like spaces in order to start the equation from this specific initial condition. Indeed, while the Dirac mass in $\mathbf{R}^d$ has (optimal) regularity index $-d$ in Hölder spaces of distributions, it also belongs to $\mathcal{B}^{-d+d/p}_{p,\infty}$ for all $p \in [1,\infty]$: this improved regularity makes the analysis of the PDE much simpler when working in Besov spaces.

Another motivation comes from Malliavin calculus. Indeed, for proving Malliavin differentiability of the solution of an SPDE, one first constructs the solution of the equation driven by a noise ξ shifted in the directions of its Cameron–Martin space – typically an L^2 space. To that end, one can enlarge the regularity structure to include abstract monomials associated to the shift: we point out the recent work [3] of Cannizzaro, Friz and Gassiat on the generalised parabolic Anderson model in dimension 2. In the case where the SPDE is additive in the noise, an alternative approach consists in lifting the shift into the polynomial regularity structure: the natural framework would then be given by $\mathcal{B}_{2,2}$-type spaces of modelled distributions.

We also mention the very recent work of Prömel and Teichmann [13] where the analytical framework of the theory of regularity structures is adapted to $\mathcal{B}^{\gamma}_{p,p}$-type spaces. We now present in more details the definitions and results obtained in the present article.

Although there is no canonical choice for the space of *modelled distributions* $\mathcal{D}^{\gamma}_{p,q}$ that would mimic the Besov space $\mathcal{B}^{\gamma}_{p,q}$, we opt for a definition as close as possible – at least formally – to the definition of classical Besov spaces via differences, see Definition 2.10.

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