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A theorem of Brown–Halmos type on the Bergman space modulo finite rank operators ☆

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A B S T R A C T

In this paper, we study the product problem of Toeplitz operators on the Bergman space of the unit disk. We characterize when the product of two Toeplitz operators T_fT_g is a finite rank perturbation of another Toeplitz operator T_h, with f, g bounded harmonic and h in C^2 class with invariant Laplacian in L^1. As a consequence, we show that there is no nontrivial rank one perturbation. However, in the case rank $m \geq 2$, we construct an example that shows there are bounded harmonic functions f, g and h such that $T_fT_g - T_h$ has rank exactly m.

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1. Introduction

Many algebraic properties of Toeplitz operators on analytic function spaces have been studied. We are concerned with the problem of when the product of two Toeplitz operators $T_f T_g$ is a finite perturbation of another Toeplitz operator T_h. In this paper, we take the Bergman space as the domain and study the question for f, g bounded harmonic and h in C^2 class with the invariant Laplacian in L^1.

1.1. Definitions

Let dA denote the Lebesgue area measure on the unit disk D in the complex plane, normalized so that the measure of the disk D is 1. The Bergman space L_a^2 is the Hilbert space consisting of analytic functions on D that are square integrable with respect to the measure dA. For $\varphi \in L^2(D, dA)$, the Toeplitz operator T_φ with symbol φ is defined densely on L_a^2 by

$$T_\varphi f = P(\varphi f),$$

where P is the orthogonal projection from $L^2(D, dA)$ to L_a^2.

For general operator S on a Hilbert space, the $rank(S)$ is defined as the dimension of closure of the range of S. S is called finite rank operator with rank r if it is bounded and $rank(S) = r < \infty$. On the Bergman space, the rank r operator has the expression

$$S = \sum_{i=1}^{r} x_i \otimes y_i,$$

where $\{x_i\}_{i=1}^r, \{y_i\}_{i=1}^r$ are two sets of linearly independent functions in L_a^2 and we use the standard notation for rank-one operators in the Hilbert space: $x \otimes y$: $h \to \langle h, y\rangle x$.

A tool that arises in the study of the Bergman space is the Berezin transform. Given an (possibly unbounded) operator S on L_a^2, with its domain containing all the normalized reproducing kernels $k_z(w) = \frac{(1-|z|^2)^2}{(1-\bar{z}w)^2}$, the Berezin transform of S is the function

$$B[S](z) = \langle Sk_z, k_z\rangle, \quad z \in D,$$

where $\langle , \rangle$ is the inner product in L_a^2. It was proved that the Berezin transform is injective [18], which means $B[S](z) = B[T](z)$ will imply $S = T$ for two operators S, T on L_a^2.

For an integrable function f on D, the Berezin transform of f is the function

$$B[f](z) = \langle f k_z, k_z\rangle$$

For $u \in L^1(D)$, it was shown in [4,12] that $B(u) = u$ if and only if u is harmonic. We shall denote the Laplacian by $\Delta = \frac{\partial^2}{\partial z \partial \bar{z}}$ and the invariant Laplacian by $\tilde{\Delta} = (1-|z|^2)^2\Delta$.