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Classification of quasi-homogeneous holomorphic curves and operators in the Cowen–Douglas class ☆

Chunlan Jiang [a], Kui Ji [a,*], Gadadhar Misra [b]

[a] Department of Mathematics, Hebei Normal University, Shijiazhuang, Hebei 050016, China
[b] Department of Mathematics, Indian Institute of Science, Bangalore, 560 012, India

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ABSTRACT

In this paper we study quasi-homogeneous operators, which include the homogeneous operators, in the Cowen–Douglas class. We give two separate theorems describing canonical models (with respect to equivalence under unitary and invertible operators, respectively) for these operators using techniques from complex geometry. This considerably extends the similarity and unitary classification of homogeneous operators in the Cowen–Douglas class obtained recently by the last author and A. Korányi. In a significant generalization of the properties of the homogeneous operators, we show that quasi-homogeneous operators are irreducible and determine which of them are strongly irreducible. Applications include the equality of the topological and algebraic K-group of a quasi-homogeneous operator and an affirmative answer to a well-known question of Halmos.

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* Corresponding author.
E-mail addresses: cljiang@hebtu.edu.cn (C. Jiang), jikui@hebtu.edu.cn (K. Ji), gm@math.iisc.ernet.in (G. Misra).

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1. Introduction

For a plane domain Ω, in the paper [3], Cowen and Douglas introduced an important class of operators $B_n(\Omega)$. It was shown by them that for operators T in $B_n(\Omega)$, the local geometry of the corresponding vector bundle E_T of rank n (curvature tensor and its higher derivatives) yields a complete set of unitary invariants for the operator T. But a tractable set of unitary (or similarity) invariants has not been found yet. The analysis of holomorphic Hermitian vector bundles in case $n > 1$ is much more complicated, see [18, Example 2.1].

In the papers [8,9], a class $\mathcal{F}B_n(\Omega)$ of operators in the Cowen–Douglas class possessing a flag structure was isolated. A complete set of unitary invariants for this class of operators were listed. Recently, Jiang and Ji have introduced methods from K-theory to classify flags of holomorphic curves in the Grassmannian in order to reduce the questions involving operators in $B_n(\Omega)$ to the case of $n = 1$ (cf. [10,11]). On the other hand, the classification of homogeneous holomorphic Hermitian vector bundles over the unit disc has been completed recently (cf. [16]) using tools from representation theory of semi-simple Lie groups. Although not complete, a similar classification over an arbitrary bounded symmetric domain is currently under way [17,19].

The methods of K-theory developed in [10,11] together with the methods of [9] makes it possible to study a much larger class of "quasi-homogeneous" operators, where the techniques from representation theory are no longer available. These methods, applied to the class of "quasi-homogeneous" operators leads to a unitary classification. In addition the bundle maps describing the triangular decomposition of Jiang and Ji have an explicit realization in terms of the inherent harmonic analysis. A model for these operators is described explicitly, which shows, among other things, that the well-known Halmos problem for the class of "quasi-homogeneous" operators has an affirmative answer.

Prompted by these results, one might imagine that the multi-variate case (replacing the planar domain Ω by the unit ball or a bounded symmetric domain) may also be accessible to these new techniques.

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For an open connected subset Ω of the complex plane $\mathbb{C}$, and $n \in \mathbb{N}$, Cowen and Douglas introduced the class of operators $B_n(\Omega)$ in their very influential paper [3]. An operator T acting on a Hilbert space $\mathcal{H}$ belongs to this class if each $w \in \Omega$, is an eigenvalue of the operator T of constant multiplicity n, these eigenvectors span the Hilbert space $\mathcal{H}$ and the operator $T - w,\ w \in \Omega$, is surjective. They showed that for an operator T in $B_n(\Omega)$, there exists a holomorphic choice of n linearly independent eigenvectors, that is, the map $w \to \ker(T - w)$ is holomorphic. Thus $\pi : E_T \to \Omega$, where

$$E_T = \{\ker(T - w) : w \in \Omega, \pi(\ker(T - w)) = w\}$$

defines a Hermitian holomorphic vector bundle on Ω.