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Principle of local reflexivity respecting nests of subspaces and the nest approximation properties

Eve Oja [a,b,*], Silja Veidenberg [a]

[a] *Institute of Mathematics and Statistics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia*
[b] *Estonian Academy of Sciences, Kohtu 6, 10130 Tallinn, Estonia*

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ABSTRACT

Versions of the principle of local reflexivity which respect given nests of subspaces of a Banach space are established. As an application, some duality and lifting theorems on approximation properties of pairs are extended to the context of nest approximation properties of Banach spaces.

1. Introduction

In July 2014, the "Aleksander Pełczyński Memorial Conference" was held in Będlewo, Poland. Bill Johnson delivered the lecture "Olek's last paper", which is the seminal

* Corresponding author.
E-mail addresses: eve.oja@ut.ee (E. Oja), silja.veidenberg@ut.ee (S. Veidenberg).

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paper [8]. He, among other things, described results, obtained together with Tadek Figiel, on the approximation properties (APs) which respect nests of subspaces (see [6,7]). These are useful extensions of the APs of pairs, introduced and studied by Figiel, Johnson, and Pełczyński in [8]. With the aim to study the APs of pairs, the first-named author of the present paper had just established versions of the principle of local reflexivity (PLR) which respect subspaces [18]. And Bill Johnson asked her whether there are versions of the PLR which respect nests of subspaces, so that they could be applied to the nest APs? The present paper is an answer to Bill's question.

We establish some versions of the PLR that respect given nests of subspaces; see Theorems 4.1, 4.2 and 4.4 in Section 4. These theorems will be based on Lemma 3.5, our main PLR lemma, which is proved in Section 3. The main PLR lemma, in its turn, will essentially use (through Lemma 3.6) a rather far-reaching extension of the Ringrose theorem (see Theorem 2.2), which is established in Section 2. Section 2 also recalls the Ringrose theorem (see Theorem 2.1) together with a necessary background on nests of subspaces.

The terminology on the nest APs is recalled in the beginning of Section 5. In Section 5, Theorem 4.2 is applied to establish duality theorems for the nest APs (see Theorems 5.1 and 5.4). These extend duality results on the APs of pairs from [15,20]. Theorem 5.4 and criteria of the nest APs, due to Figiel and Johnson [7], are applied to obtain criteria of the *duality nest* APs (see Theorem 5.10 and Corollary 5.11). In the final Section 6, since the nest APs form a special class of the convex APs, Theorem 4.4 enables us to use general lifting theorems from our recent papers [20] and [21]. We establish some lifting results and their converses for nest APs (see, e.g., Theorems 6.2, 6.7, 6.8, 6.10, and 6.13), most of them presented in a more general context of nest APs defined by an arbitrary operator ideal $\mathcal{A}$.

If $\mathcal{N}$ is a nest of subspaces of a Banach space X (the definition is recalled in Section 2), then

$$\mathcal{N}^\perp := \{Y^\perp : Y \in \mathcal{N}\}$$

and $\mathcal{N}^{\perp\perp} := (\mathcal{N}^\perp)^\perp$. Our other notation is standard. Let X and Y be Banach spaces, both real or both complex. We denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y and by $\mathcal{F}(X,Y)$ its subspace of finite-rank operators. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$ and $\mathcal{F}(X)$ for $\mathcal{F}(X,X)$. The range of an operator $S : X \to Y$ is denoted by $\operatorname{ran} S := \{Sx : x \in X\}$. The identity operator on X is denoted by I_X, and the unit sphere of X is denoted by S_X.

2. Extension of the Ringrose theorem on nests

Recall that a family of subspaces of a given Banach space is a *nest* if it is linearly ordered by inclusion. Let X be a Banach space. Let $\mathcal{N}$ be a nest of subspaces of X containing $\{0\}$. For $Y \in \mathcal{N}$, we define the *subspace* Y_- of X as follows. If $Y \neq \{0\}$, then