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Wigner's theorem on Grassmann spaces

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A R T I C L E I N F O

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A B S T R A C T

Wigner's celebrated theorem, which is particularly important in the mathematical foundations of quantum mechanics, states that every bijective transformation on the set of all rank-one projections of a complex Hilbert space which preserves the transition probability is induced by a unitary or an antiunitary operator. This vital theorem has been generalised in various ways by several scientists. In 2001, Molnár provided a natural generalisation, namely, he provided a characterisation of (not necessarily bijective) maps which act on the Grassmann space of all rank-n projections and leave the system of Jordan principal angles invariant (see [17] and [20]). In this paper we give a very natural joint generalisation of Wigner's and Molnár's theorems, namely, we prove a characterisation of all (not necessarily bijective) transformations on the Grassmann space which fix the quantity $\operatorname{Tr} PQ$ (i.e. the sum of the squares of cosines of principal angles) for every pair of rank-n projections P and Q.

1. Introduction and statement of the main result

Let H be a complex Hilbert space and I stand for the identity operator. If n is a positive integer, then we denote the set of all rank-n (self-adjoint) projections by $P_n(H)$.

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This space can be naturally identified with the *Grassmann space* of all n-dimensional subspaces of H using the map $P \mapsto \operatorname{Im} P$. In case when $n = 1$, we get the usual projective space that represents the set of all pure states of a quantum system. For $P, Q \in P_n(H)$ let us call the quantity $\operatorname{Tr} PQ$ the *transition probability* between the two projections. If $n = 1$, then this is a commonly used notion in quantum mechanics, furthermore, $\operatorname{Tr} PQ = \cos^2 \vartheta$ where ϑ is the angle between $\operatorname{Im} P$ and $\operatorname{Im} Q$. Wigner's theorem characterises symmetry transformations of $P_1(H)$ that respect the transition probability, or equivalently, that leave the angle invariant. However, this theorem can be significantly improved, namely, we can drop the bijectivity assumption and have a similar conclusion.

Theorem 1 *(E.P. Wigner, see [28], or [9,17,26]). Let $\phi\colon P_1(H) \to P_1(H)$ be a (not necessarily bijective) transformation which satisfies*

$$\operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ \qquad (P, Q \in P_1(H)).$$

Then ϕ is induced by either a linear or a conjugate-linear isometry $V\colon H \to H$, i.e.

$$\phi(P) = VPV^* \qquad (P \in P_1(H)).$$

The above result is commonly referred to as the *optimal version of Wigner's theorem*. Various generalisations of this essential result have been provided, we only mention a few of them [2,4,5,10,12,16–18,20,22–25]. This short note is particularly concerned with *Molnár's generalisation* which we explain now. Assume that $n > 1$ and $P, Q \in P_n(H)$, then the *principal angles* between P and Q are the arcuscosines of the n largest singularvalues of PQ ([1, Exercise VII.1.10], [15, Problem 559]). The system of all principal angles is denoted by $\measuredangle(P, Q) := (\vartheta_1, \dots \vartheta_n)$ where $\frac{\pi}{2} \geq \vartheta_1 \geq \vartheta_2 \geq \cdots \geq \vartheta_n \geq 0$. The origin of the notion goes back to Jordan's work [14] and has serious applications, see e.g. [7,11,13,21,27]. Molnár proved the following.

Theorem 2 *(L. Molnár, [17,20]). Let $\dim H > n \geq 2$ and $\phi\colon P_n(H) \to P_n(H)$ be a (not necessarily bijective) transformation that satisfies*

$$\measuredangle(\phi(P), \phi(Q)) = \measuredangle(P, Q) \qquad (P, Q \in P_n(H)). \tag{1}$$

Then either ϕ is induced by a linear or a conjugate-linear isometry $V\colon H \to H$, i.e.

$$\phi(P) = VPV^* \qquad (P \in P_n(H)),$$

or we have $\dim H = 2n$ and

$$\phi(P) = I - VPV^* \qquad (P \in P_n(H)).$$

As it was revealed in a personal conversation, Molnár's original desire was to prove a more general result. Namely, note that by the two projections theorem ([3,8,10]) we