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Scaling of spectra of a class of random convolution on $\mathbb{R}$ ☆

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ABSTRACT

Given a Borel probability measure μ on $\mathbb{R}$ and a real number p. We call p a spectral eigenvalue of the measure μ if there exists a discrete set Λ such that the sets

$$E(\Lambda) := \{e^{2\pi i\lambda x} : \lambda \in \Lambda\} \quad \text{and} \quad E(p\Lambda) := \{e^{2\pi ip\lambda x} : \lambda \in \Lambda\}$$

are both orthonormal basis for Hilbert space $L^2(\mu)$. In the present paper, we determine the spectral eigenvalues of a class of random convolution on $\mathbb{R}$.

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1. Introduction

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. One fundamental problem in Fourier analysis is to find a sequence $\Lambda \subseteq \mathbb{R}^d$ such that the family of complex exponential functions $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle}\}_{\lambda\in\Lambda}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$, the space of all square-integrable functions with respect to the measure μ. In this case, the measure μ is called a *spectral measure* and Λ is called a *spectrum* for μ. We also say that (μ, Λ) is a *spectral pair*. The study on spectral measures has a long history, e.g., see [35], and has been attracted much attention after the pioneer work of Fuglede [21].

Jorgensen and Pedersen [30] initiated an investigation of spectral property of the fractal measures. They showed that the infinite Bernoulli convolution μ_ρ is a spectral measure if $\rho = 2k$ for $k \in \mathbb{N}$, and is not a spectral one if $\rho = 2k + 1$ for $k \in \mathbb{N}$. Recently, Dai [4] showed that the scales $2k$ are the only values that generate spectral Bernoulli convolutions. Actually, Jorgensen and Pedersen's example opened up a new area in researching the harmonic analysis on fractals. Meanwhile, many other interesting singular measures which admit orthonormal Fourier series have been constructed, see [1–3,8,9,13,17,19,22,23,31,36–38], etc.

It is well known that a given singular spectral measure has more than one spectrum which is not obtained by translations of each other. Hence a natural question is: can we construct all spectra for a given spectral measure? It is a very challenging question. Motivated by this question, many researchers found various method to construct spectra for a given spectral measure, see [4–6,8,9,13–15,18,31–33] and the references therein. Among these results, one basic but most important constructing method is to check that whether the scaling set of a spectrum by a real number is also a spectrum. For instance, the first spectrum for the Bernoulli convolution μ_{2k} ($k \in \mathbb{N}$) given in [30] is

$$\Lambda_0(2k, C) = \left\{ \sum_{j=1}^{m} (2k)^{j-1} c_j : c_j \in C, m \in \mathbb{N} \right\} \quad \text{with} \quad C = \left\{0, \frac{k}{2}\right\}.$$

Later on, Jorgensen et al. [26], Li [32,33] provided some conditions on the integer numbers p for the scaling set

$$p\Lambda_0(2k, C) = p\left\{ \sum_{j=1}^{m} (2k)^{j-1} c_j : c_j \in C, m \in \mathbb{N} \right\} \quad \text{with} \quad C = \left\{0, \frac{k}{2}\right\},$$

to be a spectrum for μ_{2k}. In particular, Laba and Wang [31], Dutkay and Jorgensen [15], Dutkay and Haussermann [11] studied for what digits $\{0, p\}$, with p odd, the scaling set

$$p\Lambda_0(4, C) = p\left\{ \sum_{j=1}^{m} 4^{j-1} c_j : c_j \in C, m \in \mathbb{N} \right\} \quad \text{with} \quad C = \{0, 1\}$$

is a spectrum for μ_4.