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Deterministic homogenization for fast–slow systems with chaotic noise

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A B S T R A C T

Consider a fast–slow system of ordinary differential equations of the form $\dot{x} = a(x, y) + \varepsilon^{-1}b(x, y)$, $\dot{y} = \varepsilon^{-2}g(y)$, where it is assumed that b averages to zero under the fast flow generated by g. We give conditions under which solutions x to the slow equations converge weakly to an Itô diffusion X as $\varepsilon \to 0$. The drift and diffusion coefficients of the limiting stochastic differential equation satisfied by X are given explicitly.

Our theory applies when the fast flow is Anosov or Axiom A, as well as to a large class of nonuniformly hyperbolic fast flows (including the one defined by the well-known Lorenz equations), and our main results do not require any mixing assumptions on the fast flow.

1. Introduction

Let $\{\phi_t\}_{t\geq 0}$ be a smooth, deterministic flow on a finite dimensional manifold M, with invariant ergodic probability measure μ. One should think of ϕ_t as the flow generated by an ordinary differential equation (ODE) with a chaotic invariant set $\Omega \subset M$ and μ supported on Ω. Define $y(t) = \phi_t y_0$ where the initial condition y_0 is chosen at random

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according to μ. Hence $y(t) = y(t, y_0)$ is a random variable on the probability space (Ω, μ); from here on we omit y_0 from the notation, as is conventional with random variables. Let $a, b : \mathbb{R}^d \times M \to \mathbb{R}^d$ be vector fields with suitable regularity assumptions. We are interested in the asymptotic behavior of the ODE

$$\frac{dx^{(\varepsilon)}}{dt} = \varepsilon^2 a(x^{(\varepsilon)}, y) + \varepsilon b(x^{(\varepsilon)}, y) \quad , \quad x^{(\varepsilon)}(0) = \xi$$

as $\varepsilon \to 0$ and $t \to \infty$, with $\varepsilon^2 t$ remaining fixed. The initial condition $\xi \in \mathbb{R}^d$ is assumed deterministic. Due to the dependence on y_0, we interpret $x^{(\varepsilon)}$ as a random variable on Ω taking values in the space of continuous functions $C([0,T], \mathbb{R}^d)$ for some finite $T > 0$.

To make the statement of convergence precise, we define $y_\varepsilon(t) = y(\varepsilon^{-2}t)$ and x_ε as the solution to the ODE

$$\frac{dx_\varepsilon}{dt} = a(x_\varepsilon, y_\varepsilon) + \frac{1}{\varepsilon} b(x_\varepsilon, y_\varepsilon) \quad , \quad x_\varepsilon(0) = \xi \; . \tag{1.1}$$

In particular, we arrive at this equation under the rescaling $t \mapsto t/\varepsilon^2$ and setting $x_\varepsilon(t) = x^{(\varepsilon)}(t/\varepsilon^2)$. Our aim is to identify the limiting behavior of the random variable x_ε on the space of continuous functions as $\varepsilon \to 0$.

Under certain assumptions on the fast flow ϕ_t, it is known that $x_\varepsilon \to_w X$ where X is an Itô diffusion, and where $\to_w$ denotes weak convergence of random variables on the space $C([0,T], \mathbb{R}^d)$. At an intuitive level, the a term *averages* to an ergodic mean, via a law of large numbers type effect and the b term *homogenizes* to a stochastic integral, via a central limit theorem type effect. This type of problem is often referred to as *deterministic* homogenization, since the randomness is not coming from a typical stochastic process, but rather from an ergodic dynamical system with random initial condition.

Assuming rather strong mixing conditions on ϕ_t, one can show that x_ε converges weakly to the solution X of an Itô SDE

$$dX = \tilde{a}(X)dt + \sigma(X)dB \quad , \quad X(0) = \xi \tag{1.2}$$

where B is an $\mathbb{R}^d$ valued standard Brownian motion, the drift $\tilde{a} : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$\tilde{a}^i(x) = \int_\Omega a^i(x,y) d\mu(y) + \int_0^\infty \int_\Omega b(x,y) \cdot \nabla b^i(x, \phi_t y) d\mu(y) dt \tag{1.3}$$

for all $i = 1, \ldots, d$ and the diffusion coefficient $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is given by

$$\sigma(x)\sigma^T(x) = \int_0^\infty \int_\Omega \big(b(x,y) \otimes b(x, \phi_t y) + (b(x, \phi_t y) \otimes b(x,y)\big) d\mu(y) dt \; . \tag{1.4}$$

The mixing assumptions required on ϕ_t are typically very strong. For instance, the above result follows from [22] under the assumption of phi mixing with rapidly decaying mixing