Journal of Functional Analysis 272 (2017) 4122–4130

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

A higher dimensional generalization of Hersch–Payne–Schiffer inequality for Steklov eigenvalues

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ARTICLE INFO

Article history:
Received 25 August 2015
Accepted 28 February 2017
Available online 6 March 2017
Communicated by Cédric Villani

MSC:
primary 35P15
secondary 58J32

Keywords:
Differential form
Steklov eigenvalue
Hodge–Laplacian

ABSTRACT

In this paper, we generalize the Hersch–Payne–Schiffer inequality for Steklov eigenvalues to higher dimensional case by extending the trick used by Hersch, Payne and Schiffer to higher dimensional manifolds. More precisely, we show that, for a compact oriented Riemannian manifold with boundary of dimension n, the multiplication of a Steklov eigenvalue for functions and a Steklov eigenvalue for differential $(n-2)$-forms can be controlled from above by a certain eigenvalue of the Laplacian operator on the boundary.

1. Introduction

Let (M^n, g) be a compact connected Riemannian manifold with boundary. The Dirichlet-to-Neumann map $S : C^\infty(\partial M) \to C^\infty(\partial M)$ for functions is defined as

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[1] Research partially supported by a supporting project from the Department of Education of Guangdong Province with contract no. Yq2013073, and NSFC 11571215.

http://dx.doi.org/10.1016/j.jfa.2017.02.023

$$S(u) = \frac{\partial \hat{u}}{\partial \nu},$$

where the function $\hat{u}$ on M is the harmonic extension of $u \in C^\infty(\partial M)$ and ν is the outward normal vector field on ∂M. It was shown in [12] that S is a first order nonnegative self-adjoint elliptic pseudo-differential operator. So the spectrum of S is discrete and can be arranged in increasing order (counting multiplicities) as:

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq \cdots .$$

The σ_k's are called Steklov eigenvalues of (M^n, g). Steklov eigenvalues have been intensively studied recently. The overview paper [3] is an excellent survey for recent progresses of the topic.

In 1974, Hersch,Payne and Schiffer [5] proved the following inequality:

$$\sigma_p(\Omega)\sigma_q(\Omega)L(\partial\Omega)^2 \leq \begin{cases} (p+q-1)^2\pi^2 & \text{if } p+q \text{ is odd} \\ (p+q)^2\pi^2 & \text{if } p+q \text{ is even} \end{cases} \tag{1.1}$$

by an elegant trick using conjugate harmonic functions. Here Ω is a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary and $L(\partial\Omega)$ is the length of $\partial\Omega$. In [5], Hersch, Payne and Schiffer also obtained similar inequalities for multiply connected planar domains.

The Hersch–Payne–Schiffer inequality was recently generalized to surfaces of higher genus by Girouard and Polterovich in [4] via different techniques using the Ahlfors map. In fact, they proved:

$$\sigma_p(\Omega)\sigma_q(\Omega)L(\partial\Omega)^2 \leq \begin{cases} (p+q-1)^2(\gamma+l)^2\pi^2 & \text{if } p+q \text{ is odd} \\ (p+q)^2(\gamma+l)^2\pi^2 & \text{if } p+q \text{ is even.} \end{cases} \tag{1.2}$$

Here Ω is a compact oriented surface with genus γ and l connected boundary components. When $\gamma = 0$ and $l = 1$, it recovers the Hersch–Payne–Schiffer inequality for bounded simply connected planar domains. However, when $\gamma = 0$ and $l > 1$, (1.2) is different from the Hersch–Payne–Schiffer inequality for multiply connected planar domains in [5].

In [9], Raulot and Savo introduced the following Dirichlet-to-Neumann map for differential forms. Let ω be a differential r-form on ∂M, the Dirichlet-to-Neumann map $S^{(r)}$ for differential r-forms is defined as

$$S^{(r)}(\omega) = i_\nu d\hat{\omega}.$$

Here the differential r-form $\hat{\omega}$ on M is the tangential harmonic extension of ω. It is shown in [9] that $S^{(r)}$ is also a first order nonnegative self-adjoint elliptic pseudo-differential operator. So, the spectrum of $S^{(r)}$ is discrete. We can arrange the eigenvalues of $S^{(r)}$ in increasing order (counting multiplicities) as: