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A direct method of moving spheres on fractional order equations

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A B S T R A C T

In this paper, we introduce a direct method of moving spheres for the fractional Laplacian $(-\triangle)^{\alpha/2}$ with $0 < \alpha < 2$, in which a key ingredient is the narrow region maximum principle. As immediate applications, we classify non-negative solutions for semilinear equations involving the fractional Laplacian in $\mathbb{R}^n$; we prove a non-existence result for the prescribing Q_α curvature equation on $\mathbb{S}^n$; then by combining the direct method of moving planes and moving spheres, we establish a Liouville type theorem on a half Euclidean space. We expect to see more applications of this method to many other nonlinear equations involving non-local operators.

Contents

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1. Introduction

Recently, the fractional Laplacian has seen more and more applications in Physics, Chemistry, Biology, Probability, and Finance; and it has drawn more and more attention from the mathematical community. This fractional Laplacian is a pseudo-differential operator defined by

$$\begin{aligned}(-\triangle)^{\alpha/2}u(x) &\equiv C_{n,\alpha}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy \\ &\equiv C_{n,\alpha}\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy, \qquad (1.1)\end{aligned}$$

for any real number $0<\alpha<2$.

Let

$$L_\alpha \equiv \Big\{u:\mathbb{R}^n\to\mathbb{R}^1 \Big| \int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+\alpha}}dx<\infty\Big\}. \qquad (1.2)$$

Then the operator $(-\triangle)^{\alpha/2}$ is well defined on the functions u in $L_\alpha\cap C^{1,1}_{loc}$. One can see from the definition (1.1) that it is nonlocal: Even u is identically zero in a neighborhood of a point x, $(-\triangle)^{\alpha/2}u(x)$ still may not vanish. Hence, traditional methods on local differential operators, such as on Laplacian $-\triangle$ may not work on this nonlocal operator. To circumvent this difficulty, Caffarelli and Silvestre [2] introduced the *extension method* that reduced this nonlocal problem into a local one in higher dimensions. For a function $u:\mathbb{R}^n\to\mathbb{R}$, let $U:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be its extension satisfying

$$\begin{cases} div(y^{1-\alpha}\nabla U)=0, & (x,y)\in\mathbb{R}^n\times[0,\infty),\\ U(x,0)=u(x), & x\in\mathbb{R}^n.\end{cases}$$

Then

$$(-\triangle)^{\alpha/2}u(x) = -C_{n,\alpha}\lim_{y\to 0^+} y^{1-\alpha}\frac{\partial U}{\partial y}, \quad x\in\mathbb{R}^n.$$