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Non-commutativity of the exponential spectrum

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In a Banach algebra, the spectrum satisfies $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ for each pair of elements a, b. We show that this is no longer true for the exponential spectrum, thereby solving a problem open since 1992. Our proof depends on the fact that the homotopy group $\pi_4(GL_2(\mathbb{C}))$ is non-trivial.

1. Introduction

Let A be a complex, unital Banach algebra. We denote by $\mathrm{Inv}(A)$ the set of invertible elements of A. According to a standard result [1, Theorem 2.14], the connected component of $\mathrm{Inv}(A)$ containing the identity is equal to the set

$$\mathrm{Exp}(A) := \{e^{a_1}e^{a_2}\cdots e^{a_n} : a_1, \dots, a_n \in A,\ n \geq 1\}.$$

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$\operatorname{Exp}(A)$ is a normal subgroup of $\operatorname{Inv}(A)$, and the quotient $\operatorname{Inv}(A)/\operatorname{Exp}(A)$ is called the *index group* of A.

The *spectrum* $\sigma(a)$ and *exponential spectrum* $\epsilon(a)$ of an element $a \in A$ are defined by

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \operatorname{Inv}(A)\},$$
$$\epsilon(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin \operatorname{Exp}(A)\}.$$

They are compact sets, related by the inclusions $\partial\epsilon(a) \subset \sigma(a) \subset \epsilon(a)$. Thus $\epsilon(a)$ is obtained from $\sigma(a)$ by filling some (or possibly no) holes of $\sigma(a)$.

The exponential spectrum was introduced by Harte [4] to exploit the fact that, if $\theta : A \to B$ is a surjective continuous homomorphism of Banach algebras, then $\theta(\operatorname{Exp}(A)) = \operatorname{Exp}(B)$ (whereas the inclusion $\theta(\operatorname{Inv}(A)) \subset \operatorname{Inv}(B)$ may be strict). The exponential spectrum was also used by Murphy [7] to simplify proofs of certain results about Toeplitz operators.

Some properties of the spectrum are shared by the exponential spectrum, whilst others are not. Most of these are straightforward to analyze and were treated long ago. However, there is an interesting exception. It is a useful result in several areas of mathematics that, for every pair of elements $a, b \in A$, we have

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}. \tag{1}$$

This follows from the elementary algebraic fact that, if $1 - ab$ has inverse u, then $1 - ba$ has inverse $1 + bua$. Murphy [7] raised the question as to whether this commutativity property for the spectrum is shared by the exponential spectrum. In other words, do we always have

$$\epsilon(ab) \setminus \{0\} = \epsilon(ba) \setminus \{0\}\ ? \tag{2}$$

Until now, this has remained an open problem.

Murphy [7] proved that (2) holds if either a or b belongs to $\overline{\operatorname{Inv}(A)}$, the closure of the set of invertible elements. A simple adaptation of his argument shows that, more generally, (2) holds whenever a or b belongs to $\overline{Z(A)\operatorname{Inv}(A)}$, where $Z(A)$ denotes the center of A. We omit the details.

Thus, in seeking a counterexample to (2), we may restrict attention to Banach algebras A satisfying

$$\operatorname{Exp}(A) \neq \operatorname{Inv}(A) \quad \text{and} \quad \overline{Z(A)\operatorname{Inv}(A)} \neq A.$$

This straightaway rules out many candidates.

Foremost among the algebras that do satisfy these requirements is the Calkin algebra, namely the quotient algebra $B(H)/K(H)$, where H is a separable, infinite-dimensional Hilbert space, and $B(H)$ and $K(H)$ denote respectively the bounded operators and the