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Nonlinear time-harmonic Maxwell equations in an anisotropic bounded medium

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ABSTRACT

We find solutions $E : \Omega \to \mathbb{R}^3$ of the problem

$$\begin{cases} \nabla \times (\mu(x)^{-1}\nabla \times E) - \omega^2\varepsilon(x)E = \partial_E F(x,E) & \text{in } \Omega \\ \nu \times E = 0 & \text{on } \partial\Omega \end{cases}$$

on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ with exterior normal $\nu : \partial\Omega \to \mathbb{R}^3$. Here $\nabla\times$ denotes the curl operator in $\mathbb{R}^3$. The equation describes the propagation of the time-harmonic electric field $\Re\{E(x)e^{i\omega t}\}$ in an anisotropic material with a magnetic permeability tensor $\mu(x) \in \mathbb{R}^{3\times 3}$ and a permittivity tensor $\varepsilon(x) \in \mathbb{R}^{3\times 3}$. The boundary conditions are those for Ω surrounded by a perfect conductor. It is required that $\mu(x)$ and $\varepsilon(x)$ are symmetric and positive definite uniformly for $x \in \Omega$, and that $\mu, \varepsilon \in L^\infty(\Omega, \mathbb{R}^{3\times 3})$. The nonlinearity $F : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is superquadratic and subcritical in E, the model nonlinearity being of Kerr-type: $F(x,E) = |\Gamma(x)E|^p$ for some $2 < p < 6$ with $\Gamma(x) \in GL(3)$ invertible for every $x \in \Omega$ and $\Gamma, \Gamma^{-1} \in L^\infty(\Omega, \mathbb{R}^{3\times 3})$. We prove the existence of a ground state solution and of bound states if F is even in E. Moreover

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if the material is uniaxial we find two types of solutions with cylindrical symmetries.

1. Introduction

The paper concerns electromagnetic waves in an anisotropic, inhomogeneous and non-linear medium Ω in the absence of charges, currents and magnetization. In such a medium the constitutive relations between the electric displacement field $\mathcal{D}$ and the electric field $\mathcal{E}$ as well as between the magnetic induction $\mathcal{H}$ and the magnetic field $\mathcal{B}$ are given by

$$\mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P}_{NL} \quad \text{and} \quad \mathcal{B} = \mu\mathcal{H},$$

where ε is the (linear) permittivity tensor of the anisotropic material, and $\mathcal{P}_{NL}$ stands for the nonlinear polarization. In anisotropic and inhomogeneous media ε depends on $x \in \Omega$, and $\mathcal{P}_{NL}$ depends on the direction of the vector $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ and on $x \in \Omega$. The permittivity tensor $\varepsilon(x) \in \mathbb{R}^{3\times3}$ and the permeability tensor $\mu(x) \in \mathbb{R}^{3\times3}$ are assumed to be symmetric and uniformly positive definite for $x \in \Omega$. The Maxwell equations

$$\begin{cases} \nabla \times \mathcal{H} = \partial_t \mathcal{D}, & \operatorname{div}(\mathcal{D}) = 0, \\ \partial_t \mathcal{B} + \nabla \times \mathcal{E} = 0, & \operatorname{div}(\mathcal{B}) = 0, \end{cases}$$

together with the constitutive relations lead to the equation (see Saleh and Teich [26])

$$\nabla \times \big(\mu(x)^{-1} \nabla \times \mathcal{E}\big) + \varepsilon \partial_t^2 \mathcal{E} = -\partial_t^2 \mathcal{P}_{NL}.$$

In the time-harmonic case the fields $\mathcal{E}$ and $\mathcal{P}$ are of the form $\mathcal{E}(x,t) = \Re\{E(x)e^{i\omega t}\}$, $\mathcal{P}_{NL}(x,t) = \Re\{P(x)e^{i\omega t}\}$, with $E(x), P(x) \in \mathbb{C}^3$, so we arrive at the time-harmonic Maxwell equation

$$\nabla \times \big(\mu(x)^{-1} \nabla \times E\big) - V(x)E = f(x,E) \qquad \text{in } \Omega, \tag{1.1}$$

where $V(x) = \omega^2 \varepsilon(x)$ and $f(x,E)$ takes care of the nonlinear polarization. We consider nonlinearities of the form $f(x,E) = \partial_E F(x,E)$. In Kerr-like media one has

$$F(x,E) = |\Gamma(x)E|^4$$

with $\Gamma(x) \in GL(3)$ invertible for every $x \in \Omega$ and $\Gamma, \Gamma^{-1} \in L^\infty(\Omega, \mathbb{R}^{3\times3})$. This will be our model nonlinearity but we shall consider more general nonlinearities; see Section 2.

The goal of this paper is to find solutions $E : \Omega \to \mathbb{R}^3$ of (1.1) together with the boundary condition

$$\nu \times E = 0 \qquad \text{on } \partial\Omega \tag{1.2}$$