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Hölder continuous solutions of Boussinesq equation with compact support

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ABSTRACT

We show the existence of Hölder continuous solution of Boussinesq equations in whole space which has compact support both in space and time.

1. Introduction

In this paper, we consider the following Boussinesq system

$$\begin{cases} v_t + \mathrm{div}(v \otimes v) + \nabla p = \theta e_2, \quad (x,t) \in R^2 \times R, \\ \mathrm{div} v = 0, \quad (x,t) \in R^2 \times R, \\ \theta_t + \mathrm{div}(v\theta) = 0, \quad (x,t) \in R^2 \times R. \end{cases} \tag{1.1}$$

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Here $e_2 = (0,1)^T$, v is the velocity vector, p is the pressure, θ is a scalar function. The Boussinesq equations arise from many geophysical flows, such as atmospheric fronts and ocean circulations (see, for example, [16,18]). To understand the turbulence phenomena in fluid mechanics, one needs to go beyond classical solutions. The pair (v, p, θ) on $R^2 \times R$ is called a weak solution of (1.1) if they belong to $L^2_{loc}(R^2 \times R)$ and solve (1.1) in the following sense:

$$\int\limits_R \int\limits_{R^2} (\partial_t \varphi \cdot v + v \otimes v : \nabla \varphi + p \mathrm{div} \varphi + \theta e_2 \cdot \varphi) dx dt = 0,$$

for all $\varphi \in C_c^\infty(R^2 \times R; R^2)$.

$$\int\limits_R \int\limits_{R^2} (\partial_t \phi \theta + v \cdot \nabla \phi \theta) dx dt = 0,$$

for all $\phi \in C_c^\infty(R^2 \times R; R)$ and

$$\int\limits_R \int\limits_{R^2} v \cdot \nabla \psi dx dt = 0,$$

for all $\psi \in C_c^\infty(R^2 \times R; R)$.

The study of weak solutions in fluid dynamics attracts more and more peoples interests. One of the famous problem is the Onsager conjecture on Euler equation which says that the incompressible Euler equation admits Hölder continuous weak solution which dissipates kinetic energy. More precisely, the Onsager conjecture on Euler equation can be stated as following:

(1) $C^{0,\alpha}$ solutions are energy conservative when $\alpha > \frac{1}{3}$.
(2) For any $\alpha < \frac{1}{3}$, there exist dissipative solutions with $C^{0,\alpha}$ regularity.

The part (1) has been proved by P. Constantin, E. Weinan and E. Titi in [7] and also by P. Constantin, etc. in [5] with slightly weaker assumption.

The part (2) seems more subtle and has been treated by many authors. For weak solutions, the non-uniqueness results have been obtained by V. Scheffer [19], A. Shnirelman [21,20] and Camillo De Lellis, László Székelyhidi [22,9]. In particular, a great progress in the construction of Hölder continuous solution was made by Camillo De Lellis, László Székelyhidi etc. in recent years. In fact, Camillo De Lellis and László Székelyhidi developed an iterative scheme in [10], together with the aid of Beltrami flow on T^3 and Geometric Lemma, and constructed a continuous periodic solution which satisfies the prescribed kinetic energy. The solution is a superposition of infinitely many weakly interacting Beltrami flows. Building on the iterative techniques in [10] and Nash–Moser mollify techniques, they constructed Hölder continuous periodic solutions with exponent $\theta < \frac{1}{10}$,