Journal of Functional Analysis 273 (2017) 2070–2106

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

A bridge between Sobolev and Escobar inequalities and beyond

Francesco Maggi [a,*], Robin Neumayer [b]

[a] *Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, I-34151, Trieste, Italy* [1]

[b] *Department of Mathematics, The University of Texas at Austin, 2515 Speedway Stop C1200, Austin, TX 78712, USA*

A R T I C L E I N F O

Article history:
Received 8 September 2016
Accepted 13 June 2017
Available online 20 June 2017
Communicated by H. Brezis

Keywords:
Sobolev inequality
Escobar inequality
Mass transportation
Yamabe problem

A B S T R A C T

The classical Sobolev and Escobar inequalities are embedded into the same one-parameter family of sharp trace-Sobolev inequalities on half-spaces. Equality cases are characterized for each inequality in this family by tweaking a well-known mass transportation argument and lead to a new comparison theorem for trace Sobolev inequalities. The case $p = 2$ corresponds to a family of variational problems on conformally flat metrics which was previously settled by Carlen and Loss with their method of competing symmetries. In this case minimizers interpolate between conformally flat spherical and hyperbolic geometries, passing through the Euclidean geometry defined by the fundamental solution of the Laplacian.

* Corresponding author.
E-mail addresses: fmaggi@ictp.it (F. Maggi), rneumayer@math.utexas.edu (R. Neumayer).

[1] On leave from the University of Texas at Austin.

http://dx.doi.org/10.1016/j.jfa.2017.06.013

1. Introduction

1.1. A variational problem interpolating the Sobolev and Escobar inequalities

The goal of this paper is to illustrate a strong link between the *Sobolev inequality* on $\mathbb{R}^n$

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \geq S\,\|u\|_{L^{p^\star}(\mathbb{R}^n)} \qquad p^\star = \frac{np}{n-p}\,, \tag{1.1}$$

and the *Escobar inequality* on the half-space $H = \{x_1 > 0\}$

$$\|\nabla u\|_{L^p(H)} \geq E\,\|u\|_{L^{p^\#}(\partial H)} \qquad p^\# = \frac{(n-1)p}{n-p}\,, \tag{1.2}$$

where $n \geq 2$, $p \in [1, n)$, and S and E denote the optimal constants. These classical sharp inequalities both arise as particular cases of the variational problem

$$\begin{aligned}\Phi(T) &= \Phi^{(p)}(T)\\ &= \inf\Big\{\|\nabla u\|_{L^p(H)} : \|u\|_{L^{p^\star}(H)} = 1\,, \|u\|_{L^{p^\#}(\partial H)} = T\Big\} \qquad T \geq 0\,, \end{aligned}\tag{1.3}$$

with $T = 0$ in the case of (1.1), and with $T = T_E$ for a suitable $T_E > 0$ in the case of (1.2). Our main result (consisting of Theorems 1.1 and 1.2 below) characterizes the minimizers of $\Phi(T)$ for every $T > 0$ and allows one to describe the behavior of $\Phi(T)$ for every $T > 0$.

The cases $p = 2$ and $p = 1$ have interpretations in conformal geometry and in capillarity theory respectively. In particular, when $p = 2$, (1.3) amounts to minimizing a total curvature functional among conformally flat metrics on H – see (1.26) below. An interesting feature of this problem is that the corresponding minimizing geometries change their character from spherical (for $T \in (0, T_E)$) to hyperbolic (for $T > T_E$).

The characterization of minimizers in (1.3) when $p = 2$ is due to Carlen and Loss. In [5] they deduce this result by an elegant application of their method of competing symmetries. The competing symmetries in play are: (1) the operation of spherical decreasing rearrangement, and (2) the composition of a pull-back by inverse stereographic projection from $\mathbb{R}^n$ to the n-dimensional sphere $\mathbb{S}^n$, a rotation on $\mathbb{S}^n$, and a final push-forward by stereographic projection. The use of conformal invariance seems to pose a non-trivial obstacle to the applicability of this approach when $p \neq 2$. From this point of view, our use of mass transportation for extending the Carlen–Loss result to the full range $p \in (1, n)$ seems appropriate.

Let us start by setting our terminology and framework, focusing on the case $p \in (1, n)$. We work with locally summable functions $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ that are vanishing at infinity, that is, $|\{|u| > t\}| < \infty$ for every $t > 0$. If Du denotes the distributional gradient of u, then the minimization in (1.3) is over functions with $Du = \nabla u\, dx$ for $\nabla u \in L^p(H; \mathbb{R}^n)$. For